Schröder matrix as inverse of Delannoy matrix

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Abstract

Using Riordan arrays, we introduce a generalized Delannoy matrix by weighted Delannoy numbers. It turn out that Delannoy matrix, Pascal matrix, and Fibonacci matrix are all special cases of the generalized Delannoy matrices, meanwhile Schröder matrix and Catalan matrix also arise in involving inverses of the generalized Delannoy matrices. These connections are the focus of our paper. The half of generalized Delannoy matrix is also considered. In addition, we obtain a combinatorial interpretation for the generalized Fibonacci numbers.

Keywords: Delannoy number; Pascal matrix; Catalan number; Schröder number; Riordan array.

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1 Introduction

The Pascal matrix [2–4] appears often in combinatorics, probability and linear algebra. The infinite lower triangular Pascal matrix $\mathcal{P}$ is defined by generic term $p_{n,k} = \binom{n}{k}$, where the binomial coefficient $\binom{n}{k}$ counts the number of lattice paths from $(0,0)$ to $(n-k,k)$ with steps $(1,0)$ and $(0,1)$, which satisfy the recurrence relation

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

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It is easy to check that the inverse of $\mathcal{P}$ is $\mathcal{P}^{-1} = \left((-1)^{n-k}\binom{n}{k}\right)_{n,k\geq 0}$.

The generic term $f_{n,k} = \binom{k}{n-k}$ of the Fibonacci matrix $\mathcal{F} = (f_{n,k})_{n,k\geq 0}$ counts the number of lattice paths from $(0,0)$ to $(n-k,k)$ with steps $(0,1)$ and $(1,1)$, and the entries of the Fibonacci matrix $\mathcal{F}$ satisfy the recurrence relation

$$f_{n+1,k+1} = f_{n,k} + f_{n-1,k}.$$

The first few rows of $\mathcal{F}$ and $\mathcal{F}^{-1}$ are:

$$\mathcal{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \mathcal{F}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 2 & -2 & 1 & 0 & \cdots \\ 0 & -5 & 5 & -3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of $\mathcal{F}$ are the Fibonacci numbers defined by ordinary generating function $\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2}$. In this inverse $\mathcal{F}^{-1}$, if we ignore the signs, we find that the row sums are the Catalan numbers C_n, which are defined by ordinary generating function $\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$.

The Delannoy number $d(n,k)$ may be defined as the number of lattice paths from $(0,0)$ to (n,k) with steps $(1,0)$, $(0,1)$, and $(1,1)$. If we introduce the infinite lower triangular Delannoy matrix $\mathcal{D} = (d_{n,k})_{n,k\geq 0}$ by $d_{n,k} = d(n-k,k)$. Then its entries satisfy the recurrence relation

$$d_{n+1,k+1} = d_{n,k+1} + d_{n,k} + d_{n-1,k},$$

and $d_{n,k}$ counts the number of lattice paths from $(0,0)$ to $(n-k,k)$ with steps $(1,0)$, $(0,1)$ and $(1,1)$. The first few entries of $\mathcal{D}$ and $\mathcal{D}^{-1}$ are as follows:

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 5 & 5 & 1 & 0 & 0 & \cdots \\ 1 & 7 & 13 & 7 & 1 & 0 & \cdots \\ 1 & 9 & 25 & 25 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \mathcal{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -3 & 1 & 0 & 0 & 0 & \cdots \\ -6 & 10 & -5 & 1 & 0 & 0 & \cdots \\ 22 & -38 & 22 & -7 & 1 & 0 & \cdots \\ -90 & 158 & -98 & 38 & -9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

An immediate calculation show that the row sums of the Delannoy matrix $\mathcal{D}$ are the Pell sequence $1, 2, 5, 12, \cdots$, while the row sums of unsigned entries of $\mathcal{D}^{-1}$ are the large Schröder numbers $1, 2, 6, 22, \cdots$, see [7, 19].

The analogue between the Pascal matrix, Fibonacci matrix, and Delannoy matrix motivate us to study a more generalized situation. In this paper, by using Riordan arrays, we introduce a generalized Delannoy matrices by weighted Delannoy numbers. It turns out that Delannoy matrix, Pascal matrix, and Fibonacci matrix are all special cases of generalized Delannoy matrices, meanwhile Schröder matrix and Catalan matrix also arise in involving inverses of weighted Delannoy matrices. These connections are the focus of our paper. The half of generalized Delannoy matrix is also considered. In addition, we obtain a combinatorial interpretation for the generalized Fibonacci numbers.

2 Riordan arrays

Riordan arrays were first introduced in 1991 by Shapiro et al. [16], and many works and applications on this subject have been done, for example [5,6,8,9,17]. An infinite lower triangular matrix $\mathcal{D}$ is called a Riordan array if its column k ($k = 0, 1, 2, \cdots$) has generating function $g(x)f(x)^k$, where $g(x) = \sum_{n=0}^{\infty} g_n x^n$ and $f(x) = \sum_{n=0}^{\infty} f_n x^n$ are formal power series with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. That is, the general term of array $\mathcal{D}$ is $d_{n,k} = [x^n]g(x)f(x)^k$, where $[x^n]h(x)$ denotes the coefficient of x^n in power series $h(x)$.

Suppose we multiply the array $\mathcal{D} = (g(x), f(x))$ by a column vector $(b_0, b_1, b_2, \ldots)^T$ and get a column vector $(a_0, a_1, a_2, \ldots)^T$. Let $b(x)$ be the ordinary generating functions for the sequence $(b_0, b_1, b_2, \ldots)^T$. Then it follows that the ordinary generating functions for the sequence $(a_0, a_1, a_2, \ldots)^T$ is $g(x)b(f(x))$. If we identify a sequence with its ordinary generating function, the composition rule can be presented as

$$(g(x), f(x))b(x) = g(x)b(f(x)). \tag{1}$$

This is called the fundamental theorem for Riordan arrays and this leads to the

multiplication rule for the Riordan arrays (see Shapiro et al [16]):

$$(g(x), f(x))(h(x), l(x)) = (g(x)h(f(x)), l(f(x))). \tag{2}$$

The inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = (1/g(\bar{f}(x)), \bar{f}(x)), \tag{3}$$

where $\bar{f}(x)$ is the compositional inverse of $f(x)$.

The bivariate generating function $D(x, y)$ of the Riordan array $\mathcal{D} = (g(x), f(x))$ is given by

$$D(x, y) = (g(x), f(x))\frac{1}{1 - yx} = \frac{g(x)}{1 - yf(x)}. \tag{4}$$

Lemma 2.1 ([8]). Let $\mathcal{D} = (d_{n,k})$ be an infinite lower triangular matrix. Then $\mathcal{D}$ is a Riordan array if and only if $d_{0,0} = 1$ and there exists two sequences $A = (a_i)_{i\geq 0}$ and $Z = (z_i)_{i\geq 0}$ with $a_0 \neq 0$ such that

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, n, k = 0, 1, \cdots, \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots, n = 0, 1, \cdots. \end{aligned}$$

Such sequences are called the A-sequence and the Z-sequence of the Riordan array D, respectively.

Lemma 2.2([8]). Let $\mathcal{D} = (g(x), f(x))$ be a Riordan array with inverse $\mathcal{D}^{-1} = (d(x), h(x))$. Then the A- and Z-sequences of $\mathcal{D}$ are

$$A(x) = \frac{x}{h(x)};\ Z(x) = \frac{1}{h(x)}(1 - d(x)). \tag{5}$$

Example 2.1. (a) It is well known that the Pascal matrix $\mathcal{P} = \left(\binom{n}{k}\right)$ can be expressed as the Riordan array $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$, and the generating functions of its A- and Z-sequences are $A(x) = 1 + x$, $Z(x) = 1$. More generally, for the generalized Pascal array $\mathcal{P}[r] = \left(r^{n-k}\binom{n}{k}\right)$, we have $\mathcal{P}[r] = \left(\frac{1}{1-rx}, \frac{x}{1-rx}\right)$, $\mathcal{P}[r]^{-1} = \left(\frac{1}{1+rx}, \frac{x}{1+rx}\right)$.

(b) The Fibonacci matrix $\mathcal{F} = \left(\binom{k}{n-k}\right)$ can be expressed as the Riordan matrix $\mathcal{F} = (1, x + x^2)$, and its inverse is $\mathcal{F}^{-1} = (1, xC(-x))$.

The set of all Riordan arrays associated with the usual row-by-column product shown in (2) forms a group denoted by $\mathscr{R}$, where $I = (1,\, x)$ acts as the identity for this product, that is, $(1,\, x) * (d(x),\, h(x)) = (d(x),\, h(x)) * (1, x) = (d(x),\, h(x))$. A subgroup, denoted by $\mathscr{B}$, of $\mathscr{R}$ is the set of of *Bell-type arrays* or *renewal arrays*, that is the Riordan arrays $D = (d(x),\, h(x))$ for which $h(x) = xd(x)$, which was considered in the literature [15].

He [9] uses the sequence characterization of Bell-type Riordan array to define (c, r)-(generalized or parametric) Catalan numbers with parameters c and r, which have the generating function

$$d_{c,r}(x) = \frac{1 - (c-r)x - \sqrt{1 - 2(c+r)x + (c-r)^2x^2}}{2rx}. \tag{6}$$

(6) was shown in (10) of [9]. The corresponding Bell-type Riordan arrays $(d_{c,r}(x), xd_{c,r}(x))$ are called the (c, r)-(generalized or parametric) Catalan triangles. The Taylor expansion and some properties of the generalized Catalan numbers and generalized Catalan triangles are presented. In addition, [9] gives some combinatorial interpretations for the Bell-type Riordan arrays including those of the generalized Catalan triangles. [9] also discusses the inverse of the generalized Catalan triangles, from which the expressions of the parametric Catalan numbers and triangles in terms of classical Catalan numbers are given. All of those results will be useful in Sections 3 and 4 of this paper.

3 Generalized Delannoy matrix

We consider those lattice paths in the Cartesian plane starting from $(0, 0)$ that use the steps E, D, and N, where $E = (1, 0)$, a east-step; $D = (1, 1)$, a diagonal-step; and $N = (0, 1)$, a north-step, with assigned weights e, d, and w, respectively, where e, d and w are positive integers. Many properties and applications of Delannoy numbers have been discussed [1, 10, 12, 13, 20, 21]. In combinatorics, we regard weight as the number of colors and normalize by setting $w = 1$. Let P be a path. We define the weight $w(P)$ to be the product of the weight of the steps. Let $A(n, k)$ be the set of all weighted lattice paths ending at the point $(n - k, k)$ and let $B(n, k)$ be the set

of lattice paths in $A(n,k)$ which have no east steps on the x-axis. The generalized Delannoy numbers $a_{n,k}$ are the sum of all $w(P)$ with P in $A(n,k)$ and $b_{n,k}$ are the sum of all $w(P)$ with P in $B(n,k)$.

The array $\mathcal{A}$ is called the generalized Delannoy matrix of the first kind, and the array $\mathcal{B}$ is called the generalized Delannoy matrix of the second kind. It is straight forward to show that the array $\mathcal{A} = (a_{n,k})_{n,k\geq 0}$ satisfies for $n \geq 0$ and $k \geq 0$ the recursion equation

$$a_{n+1,k+1} = ea_{n,k+1} + a_{n,k} + da_{n-1,k} \tag{7}$$

with the conditions $a_{0,0} = 1$ and $a_{n,k} = 0$ if $n < 0$ and $a_{n,k} = 0$ if $n < k$. Using this recursion equation we get the generating function $A(x,y)$ of the array $(a_{n,k})$ is

$$A(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^n y^k = \frac{1}{1-ex-xy-dx^2y}. \tag{8}$$

Similarly, the array $\mathcal{B} = (b_{n,k})_{n,k\geq 0}$ satisfies for $n \geq 0$ and $k \geq 0$ the recursion equation

$$b_{n+1,k+1} = eb_{n,k+1} + b_{n,k} + db_{n-1,k} \tag{9}$$

with the conditions $b_{0,0} = 1$, $b_{n,0} = 0$ if $n \geq 1$ and $b_{n,k} = 0$ if $n < k$. Hence we have

$$B(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{n,k}x^n y^k = \frac{1-ex}{1-ex-xy-dx^2y}. \tag{10}$$

Some entries of the arrays $\mathcal{A}$ and $\mathcal{B}$ are:

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ e & 1 & 0 & 0 & 0 & \cdots \\ e^2 & 2e+d & 1 & 0 & 0 & \cdots \\ e^3 & 3e^2+2ed & 3e+2d & 1 & 0 & \cdots \\ e^4 & 4e^3+3e^2d & 6e^2+6ed+d^2 & 4e+3d & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & e+d & 1 & 0 & 0 & \cdots \\ 0 & e^2+ed & 2e+2d & 1 & 0 & \cdots \\ 0 & e^3+e^2d & 3e^2+4ed+d^2 & 3e+3d & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 3.1. The generalized Delannoy matrices of the first kind and of the second kind can be represented by Riordan arrays as

$$\mathcal{A} = \left(\frac{1}{1-ex}, \frac{x+dx^2}{1-ex}\right), \ \mathcal{B} = \left(1, \frac{x+dx^2}{1-ex}\right).$$

Proof From (4), the bivariate generating function of the Riordan array $\left(\frac{1}{1-ex}, \frac{x+dx^2}{1-ex}\right)$ is $\left(\frac{1}{1-ex}, \frac{x+dx^2}{1-ex}\right)\frac{1}{1-yx} = \frac{1}{1-ex} \cdot \frac{1}{1-y\frac{x+dx^2}{1-ex}} = \frac{1}{1-ex-xy-dx^2y}$. Hence, the result follows by (8). In a similar way we can prove another result. □

Corollary 3.2. The general terms of the arrays $\mathcal{A}$ and $\mathcal{B}$ are given by

$$a_{i,j} = \sum_{k=0}^{i-j} \binom{j}{k} \binom{i-k}{j} e^{i-j-k} d^k, \tag{11}$$

$$b_{i,j} = \sum_{k=0}^{i-j} \binom{j}{k} \binom{i-k-1}{j-1} e^{i-j-k} d^k. \tag{12}$$

It is easy to see that after deleting the first column and the first row of $\mathcal{B}$, we obtain a Bell-type Riordan array $((1+dx)/(1-ex), (x+dx^2)/(1-ex))$. Hence, its inverse $((1+dx)/(1-ex), (x+dx^2)/(1-ex))^{-1} = (f(x;e,d), xf(x;e,d))$ is also a Bell-type Riordan array. From (30) and (10) of [9], we have

$$f(x;e,d) = \frac{1+ex-\sqrt{e^2x^2+2(e+2d)x+1}}{-2dx} = d_{-(e+d),-d}(x),$$

where $d_{c,r}(x)$ is shown in (6).

Let $A_n = \sum_{k=0}^{n} a_{n,k}$. Then A_n is the sum of the weights of all lattice paths from origin $(0,0)$ to the line $x+y=n$ using steps $(1,0)$, $(1,1)$ and $(0,1)$ with weights e, d, and 1, respectively. Setting $y=1$ in the above proof, we get the generating function for the row sums of $\mathcal{A}$ is

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{1-(1+e)x-dx^2}. \tag{13}$$

Therefore, the sequence $\{A_n\}$ satisfy the following recurrence relation :

$$A_n = (e+1)A_{n-1} + dA_{n-2}, \ n \geq 2,$$

with $A_0 = 1$ and $A_1 = e + 1$. The sequence $\{A_n\}$ is called generalized Fibonacci numbers, and by (13) its generic element is

$$A_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (e+1)^{n-2k} d^k.$$

Let $B_n = \sum_{k=0}^{n} b_{n,k}$. Then B_n is the sum of the weights of all lattice paths from origin $(0,0)$ to the line $x + y = n$ using steps $(1,0)$, $(1,1)$ and $(0,1)$ with weights e, d, and 1, respectively, and without step $(1,0)$ on the x-axis. The generating function for the row sums of $\mathcal{B}$ is

$$\sum_{n=0}^{\infty} B_n x^n = \frac{1-ex}{1-(1+e)x-dx^2}. \tag{14}$$

Therefore, the sequence $\{B_n\}$ satisfy the following recurrence relation :

$$B_n = (e+1)B_{n-1} + dB_{n-2},\ n \geq 2,$$

with $B_0 = 1$ and $B_1 = 1$. Furthermore, $B_0 = A_0$, and $B_n = A_n - eA_{n-1} = A_{n-1} + dA_{n-2}$ for $n \geq 1$.

In the case $e = 1$ and $d = 0$, we have $\mathcal{A} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$, which is the Pascal matrix $\mathcal{P}$. When $e = d = 1$, $\mathcal{A} = \left(\frac{1}{1-x}, \frac{x+x^2}{1-x}\right)$ is the Delannoy matrix $\mathcal{D}$ whose row sums are Pell numbers. When $e = 0$, and $d = 1$, $\mathcal{A} = (1, x + x^2)$ is the Fibonacci matrix $\mathcal{F}$. So we can consider the generalized Delannoy matrices $\mathcal{A}$ as an extension of the Pascal matrix, Delannoy matrix and Fibonacci matrix.

4 Generalized Schröder matrix

In this section, we consider those lattice paths from $(0,0)$ with steps $E = (1,0)$, $D = (1,1)$ and $N = (0,1)$ which are endowed with weighs d, e and $w = 1$, respectively. Let $R(n,k)$ be the set of all weighted lattice paths ending at the point $(n-k, n)$ and that its last step is not east step and that never falling below the line $y = x$. Let $S(n,k)$ be the set of lattice paths in $R(n,k)$ which have no diagonal steps on the line $y = x$. Let $r_{n,k}$ be the sum of all $w(P)$ with P in $R(n,k)$ and let $s_{n,k}$ be the

sum of all $w(P)$ with P in $S(n,k)$. Then $r_n(e,d) = r_{n,0} + dr_{n,1} + \cdots + d^n r_{n,n}$ is the sum of weights of all weighted paths ending on (n,n) that never falling below the line $y = x$, and $s_n(e,d) = s_{n,0} + ds_{n,1} + \cdots + d^n s_{n,n}$ is the sum of weights of paths with no step $(1,1)$ on the line $y = x$ and ending on (n,n) and that never falling below the line $y = x$. We call $r_n(e,d)$ the n-th large weighted Schröder number and $r_n(e,d)$ the n-th small weighted Schröder number. The array $\mathcal{R}$ is called the generalized Schröder matrix of the first kind, and the array $\mathcal{S}$ is called the generalized Schröder matrix of the second kind. Some entries of the arrays $\mathcal{R} = (r_{n,k})_{n,k\geq 0}$ and $\mathcal{S} = (s_{n,k})_{n,k\geq 0}$ are:

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ e & 1 & 0 & 0 & 0 & \cdots \\ e^2+ed & 2e+d & 1 & 0 & 0 & \cdots \\ e^3+3e^2d+2ed^2 & 3e^2+5ed+2d^2 & 3e+2d & 1 & 0 & \cdots \\ e^4+6e^3d+10e^2d^2+5ed^3 & 4e^3+14e^2d+15ed^2+5d^3 & 6e^2+11ed+5d^2 & 4e+3d & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & e+d & 1 & 0 & 0 & \cdots \\ 0 & e^2+3ed+2d^2 & 2e+2d & 1 & 0 & \cdots \\ 0 & e^3+6e^2d+10ed^2+5d^3 & 3e^2+8ed+5d^2 & 3e+3d & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By considering the positions preceding to the last step of a lattice path in $R(n,k)$, we have $r_{0,0} = 1$, $r_{0,k} = 0$ for $k > 0$ and

$$r_{n+1,k+1} = r_{n,k} + (e+d)r_{n,k+1} + (e+d)d\, r_{n,k+2} + \cdots + (e+d)d^{n-k-1} r_{n,n},\ n,k \geq 0, \quad (15)$$

$$r_{n+1,0} = er_{n,0} + edr_{n,1} + \cdots + ed^n r_{n,n},\ n \geq 0. \quad (16)$$

Similarly, the array $\mathcal{S} = (s_{n,k})_{n,k\geq 0}$ satisfies the recurrence

$$s_{n+1,k+1} = s_{n,k} + (e+d)s_{n,k+1} + (e+d)ds_{n,k+2} + \cdots + (e+d)d^{n-k-1} s_{n,n},\ n,k \geq 0, \quad (17)$$

with the conditions $s_{0,0} = 1$, $s_{n,0} = 0$ if $n \geq 1$ and $s_{0,k} = 0$ if $k \geq 1$.

If $d = 0$ and $e \neq 0$, we find that $\mathcal{R} = \left(\frac{1}{1+ex}, \frac{x}{1+ex}\right)^{-1} = \left(\frac{1}{1-ex}, \frac{x}{1-ex}\right)$, and $\mathcal{S} = \left(1, \frac{x}{1+ex}\right)^{-1} = \left(1, \frac{x}{1-ex}\right)$, which are the generalized Pascal matrices.

Theorem 4.1. If $d \neq 0$, then the array $\mathcal{R}$ has a Riordan array expression given by

$$\mathcal{R} = \left(\frac{1}{1+ex}, \frac{x-dx^2}{1+ex}\right)^{-1} = \left(1+eh(x), h(x)\right),$$

where $h(x) = \frac{1-ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2d}$.

Proof By the formulae (15) and (16), the A- and Z- sequences of the array $\mathcal{R}$ are $A = (1, (e+d), (e+d)d, \cdots)$ and $Z = (e, ed, ed^2, \cdots)$, or $A(x) = \frac{1+ex}{1-dx}$ and $Z(x) = \frac{e}{1-dx}$. Hence, form Lemmas 2.1 and 2.2, the array $\mathcal{R}$ is a Rioran array $(g(x), h(x))$ and $h(x) = x \cdot \frac{1+eh(x)}{1-dh(x)}$, $g(x) = \frac{1}{1-\frac{ex}{1-dh(x)}}$. Solving these functional equations, we obtain $h(x) = \frac{1-ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2d}$, and $g(x) = 1+eh(x)$. □

In (9) of [9], by transforming c to $e+d$ and r to d, one may obtain the following corollary.

Corollary 4.2. If $d \neq 0$, then the array $\mathcal{S}$ can be expressed as

$$\mathcal{S} = \left(1, \frac{x-dx^2}{1+ex}\right)^{-1} = \left(1, \frac{1-ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2d}\right).$$

From (28) of [9], noting $d_{n-1,k-1} = s_{n,k}$ and transforming c to $e+d$ and r to d, we have

$$s_{n,k} = \frac{k}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{2n-k-i-1}{n-k-i}e^i d^{n-i-k}.$$

Hence, we have the following relationship between $s_{n,k}$ and $r_{n,k}$.

Corollary 4.3. For the generalized Schröder matrix of the first kind $\mathcal{R} = (r_{n,k})$ and the generalized Schröder matrix of the second kind $\mathcal{S} = (s_{n,k})$, there holds

$$r_{n,k} = s_{n,k} + es_{n,k+1}.$$

Theorem 4.4. The generating function for the large weighted Schröder numbers is given by

$$R(x; e, d) = \frac{1-ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2dx}.$$

Proof By definition, $r_n(e,d) = r_{n,0} + dr_{n,1} + \cdots + d^n r_{n,n}$. Hence, $R(x;e,d) = \sum_{n=0}^{\infty} r_n(e,d)x^n = (1+eh(x), h(x))\frac{1}{1-dx} = \frac{1+eh(x)}{1-dh(x)} = \frac{h(x)}{x} = \frac{1-ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2dx}$. Here we have used the fact $h(x) = x \cdot \frac{1+eh(x)}{1-dh(x)}$. □

Theorem 4.5. The generating function for the small weighted Schröder numbers is given by

$$S(x;e,d) = \frac{1+ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2(e+d)x}, \text{ and}$$
$$(1, S(x;e,d))^{-1} = \left(1, \frac{x-(e+d)x^2}{1-ex}\right).$$

Proof By definition, $s_n(e,d) = s_{n,0} + ds_{n,1} + \cdots + d^n s_{n,n}$. Hence, $S(x;e,d) = \sum_{n=0}^{\infty} s_n(e,d)x^n = (1,h(x))\frac{1}{1-dx} = \frac{1}{1-dh(x)} = \frac{1+ex-\sqrt{e^2x^2-(2e+4d)x+1}}{2(e+d)x}$. From the statement after Theorem 3.1, we immediately know that $(1,S(x;e,d))^{-1} = \left(1, \frac{x-(e+d)x^2}{1-ex}\right)$. □

From Theorem 3.1 and Theorem 4.1, the Delannoy matrix $\mathcal{A}$ and Schröder matrix $\mathcal{R}$ are inverse each other in the view $\mathcal{A}^{-1} = \left((-1)^{n-k} r_{n,k}\right)$, and $\mathcal{R}^{-1} = \left((-1)^{n-k} a_{n,k}\right)$. Similarly, the Delannoy matrix $\mathcal{B}$ and Schröder matrix $\mathcal{S}$ are inverse each other. We state this interesting result in the following theorem.

Theorem 4.6. Let $\mathcal{M}$ denote the Riordan array $(1,-x)$, then

$$\mathcal{A}^{-1} = \mathcal{M}\mathcal{R}\mathcal{M},\ \mathcal{R}^{-1} = \mathcal{M}\mathcal{A}\mathcal{M},$$

$$\mathcal{B}^{-1} = \mathcal{M}\mathcal{S}\mathcal{M},\ \mathcal{S}^{-1} = \mathcal{M}\mathcal{B}\mathcal{M}.$$

Example 4.1. If $e = d = 1$, then $R(x;1,1) = \frac{1-x-\sqrt{1-6x+x^2}}{2x}$ is the generating function for the large Schröder numbers, and $S(x;1,1) = \frac{1+x-\sqrt{1-6x+x^2}}{4x}$ is the generating function for the small Schröder numbers. Some entries of $\mathcal{R}$ and $\mathcal{S}$ are as follows:

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 10 & 5 & 1 & 0 & 0 & \cdots \\ 22 & 38 & 22 & 7 & 1 & 0 & \cdots \\ 90 & 158 & 98 & 38 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 4 & 1 & 0 & 0 & \cdots \\ 0 & 22 & 16 & 6 & 1 & 0 & \cdots \\ 0 & 90 & 68 & 30 & 8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These are Schröder triangles discussed in [11, 14].

Example 4.2. If $e = 0$, $d \neq 0$, then $R(x; 0, d) = S(x; 0, d) = \frac{1-\sqrt{1-4dx}}{2dx} = C(dx)$, where $C(x)$ is the generating function for the numbers. Some entries of $\mathcal{R} = \mathcal{S}$ are as follows:

$$\mathcal{R} = \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & d & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2d^2 & 2d & 1 & 0 & 0 & 0 & \cdots \\ 0 & 5d^3 & 5d^2 & 3d & 1 & 0 & 0 & \cdots \\ 0 & 14d^4 & 14d^3 & 9d^2 & 4d & 1 & 0 & \cdots \\ 0 & 42d^5 & 42d^4 & 28d^3 & 14d^2 & 5d & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the Catalan triangle discussed in [18].

Example 4.3. If $e = 1$ and $d = 2$, then $R(x; 1, 2) = \frac{1-x-\sqrt{1-10x+x^2}}{4x}$, and $S(x; 1, 2) = \frac{1+x-\sqrt{1-10x+x^2}}{6x}$. Some entries of $\mathcal{R}$ and $\mathcal{S}$ are as follows:

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 15 & 21 & 7 & 1 & 0 & 0 & \cdots \\ 93 & 132 & 48 & 10 & 1 & 0 & \cdots \\ 645 & 921 & 348 & 84 & 13 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 15 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 93 & 39 & 9 & 1 & 0 & \cdots \\ 0 & 645 & 276 & 72 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 7 & 7 & 1 & 0 & 0 & \cdots \\ 1 & 10 & 22 & 10 & 1 & 0 & \cdots \\ 1 & 13 & 46 & 46 & 13 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 3 & 15 & 9 & 1 & 0 & \cdots \\ 0 & 3 & 24 & 36 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 4.4. If $e = 2$ and $d = 1$, then $R(x; 2, 1) = \frac{1-2x-\sqrt{1-8x+4x^2}}{2x}$, and

$S(x;2,1) = \frac{1+2x-\sqrt{1-8x+4x^2}}{6x}$. Some entries of $\mathcal{R}$ and $\mathcal{S}$ are as follows:

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 24 & 8 & 1 & 0 & 0 & \cdots \\ 114 & 123 & 51 & 11 & 1 & 0 & \cdots \\ 600 & 672 & 312 & 87 & 14 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 12 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 57 & 33 & 9 & 1 & 0 & \cdots \\ 0 & 300 & 186 & 63 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 8 & 16 & 8 & 1 & 0 & 0 & \cdots \\ 16 & 44 & 37 & 11 & 1 & 0 & \cdots \\ 32 & 112 & 134 & 67 & 14 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 12 & 21 & 9 & 1 & 0 & \cdots \\ 0 & 24 & 60 & 45 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

5 Half of the Delannoy matrix

Denote by $H(n,k)$ the set of all weighted lattice paths ending at the point $(n-k,n)$ and using steps $E = (1,0)$, $D = (1,1)$, and $N = (0,1)$ with weights d, e, and 1, respectively. Let $h_{n,k}$ be the sum of all $w(P)$ with P in $H(n,k)$. Recall that $A(n,k)$ is the set of all weighted lattice paths ending at the point $(n-k,k)$ and the generalized Delannoy numbers $a_{n,k}$ are the sum of all $w(P)$ with P in $A(n,k)$. Hence, $h_{n,k} = a_{2n-k,n}$, and the matrix $\mathcal{H} = (h_{n,k})_{n,k\geq 0}$ is the right-hand side of the generalized Delannoy matrix $\mathcal{A} = \left(\frac{1}{1-dx}, \frac{x+ex^2}{1-dx}\right)$:

$$\mathcal{H} = \begin{pmatrix} h_{0,0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ h_{1,0} & h_{1,1} & 0 & 0 & 0 & 0 & \cdots \\ h_{2,0} & h_{2,1} & h_{2,2} & 0 & 0 & 0 & \cdots \\ h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3} & 0 & 0 & \cdots \\ h_{4,0} & h_{4,1} & h_{4,2} & h_{4,3} & h_{4,4} & 0 & \cdots \\ h_{5,0} & h_{5,1} & h_{5,2} & h_{5,3} & h_{5,4} & h_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{0,0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_{2,1} & a_{1,1} & 0 & 0 & 0 & 0 & \cdots \\ a_{4,2} & a_{3,2} & a_{2,2} & 0 & 0 & 0 & \cdots \\ a_{6,3} & a_{5,3} & a_{4,3} & a_{3,3} & 0 & 0 & \cdots \\ a_{8,4} & a_{7,4} & a_{6,4} & a_{5,4} & a_{4,4} & 0 & \cdots \\ a_{10,5} & a_{9,5} & a_{8,5} & a_{7,5} & a_{6,5} & a_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $d = 0$, then $\mathcal{A} = (1, x + ex^2)$. Therefore, $a_{n,k} = [x^n](x + ex^2)^k = \binom{k}{n-k}e^{n-k}$, and $h_{n,k} = a_{2n-k,n} = \binom{n}{n-k}e^{n-k}$. Consequently $\mathcal{H} = \left(\frac{1}{1-ex}, \frac{x}{1-ex}\right)$. In general, we have the following results.

Theorem 5.1. The inverse of the triangle $\mathcal{H}$ is given by the Riordan array

$$\mathcal{H}^{-1} = \left(\frac{1 - 2dx - dex^2}{1 + ex}, \frac{x - dx^2}{1 + ex}\right).$$

Proof By considering the positions preceding to the last step of a lattice path in $H(n, k)$, we have $h_{0,0} = 1$, $h_{0,k} = 0$ for $k > 0$ and

$$h_{n+1,k+1} = h_{n,k}+(d+e)h_{n,k+1}+(d+e)d\,h_{n,k+2}+\cdots+(d+e)d^{n-k-1}h_{n,n},\ n, k \geq 0, \quad (18)$$

$$h_{n+1,0} = (2d+e)h_{n,0}+2(d+e)dh_{n,1}+2(d+e)d^2h_{n,2}+\cdots+2(d+e)d^nh_{n,n},\ n \geq 0. \quad (19)$$

Therefore, $\mathcal{H} = (h_{n,k})_{n,k\geq 0}$ is a Riordan array whose A- and Z-sequences are are

$$A = (1, d + e, (d + e)d, (d + e)d^2, (d + e)d^3, \cdots),$$

$$Z = (2d + e, 2(d + e)d, 2(d + e)d^2, 2(d + e)d^3, \cdots).$$

Thus, the generating function of the A- and Z-sequences are $A(x) = \frac{1+ex}{1-dx}$ and $Z(x) = \frac{2d+e+dex}{1-dx}$. By Lemma 2.2, $\mathcal{H}^{-1} = \left(\frac{1-2dx-dex^2}{1+ex}, \frac{x-dx^2}{1+ex}\right)$. □

Corollary 5.2. If $d \neq 0$, then the triangle $\mathcal{H}$ is given by the Riordan array

$$\mathcal{H} = \left(\frac{1}{\sqrt{e^2x^2 - (2e + 4d)x + 1}}, \frac{1 - ex - \sqrt{e^2x^2 - (2e + 4d)x + 1}}{2d}\right).$$

Example 5.1. If $e = 0$ and $d = 1$, then $\mathcal{H} = \left(\frac{1}{\sqrt{1-4x}}, \frac{1-\sqrt{1-4x}}{2}\right)$, $\mathcal{H}^{-1} = (1 - 2x, x - x^2)$, and $\mathcal{A} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$.

Example 5.2. If $e = 1$ and $d = 1$, then $\mathcal{H} = \left(\frac{1}{\sqrt{1-6x+x^2}}, \frac{1-x-\sqrt{1-6x+x^2}}{2}\right)$, $\mathcal{H}^{-1} = \left(\frac{1-2x-x^2}{1+x}, \frac{x-x^2}{1+x}\right)$, and $\mathcal{A} = \left(\frac{1}{1-x}, \frac{x+x^2}{1-x}\right)$.

Example 5.3. If $e = 1$ and $d = 2$, then $\mathcal{H} = \left(\frac{1}{\sqrt{1-10x+x^2}}, \frac{1-x-\sqrt{1-10x+x^2}}{4}\right)$, $\mathcal{H}^{-1} = \left(\frac{1-4x-2x^2}{1+x}, \frac{x-2x^2}{1+x}\right)$, and $\mathcal{A} = \left(\frac{1}{1-2x}, \frac{x+x^2}{1-2x}\right)$.

Example 5.4. If $e = 2$ and $d = 1$, then $\mathcal{H} = \left(\frac{1}{\sqrt{1-8x+4x^2}}, \frac{1-2x-\sqrt{1-8x+4x^2}}{2}\right)$, $\mathcal{H}^{-1} = \left(\frac{1-2x-2x^2}{1+2x}, \frac{x-x^2}{1+2x}\right)$, and $\mathcal{A} = \left(\frac{1}{1-x}, \frac{x+2x^2}{1-x}\right)$.

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