

Divisibility of Generalized Catalan Numbers and Raney Numbers

Jacob Bobrowski ^{*} Tian-Xiao He [†] and Peter J.-S. Shiue [‡]

Abstract

The Raney numbers, also called Fuss-Catalan numbers, are defined by $R_k(n, r) = r \binom{kn+r}{n} / (kn+r)$. A generalized Lobb numbers is introduced. The relationship between Raney numbers and generalized Lobb numbers and the relationship between generalized Lobb numbers and generalized Catalan numbers are given. Based on the relationships among Raney numbers, generalized Lobb numbers, and generalized Catalan numbers, we present the divisibility of a certain class of those numbers.

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1 Introduction

The Fuss-Catalan numbers or Raney numbers are numbers of the form

$$R_k(n, r) := \frac{r}{kn+r} \binom{kn+r}{n}, \quad (1)$$

which are named after N. I. Fuss and E. C. Catalan (see [5, 6, 13, 15, 17]) and initially studied by Raney in [17]. The Fuss-Catalan numbers have several combinatorial applications. They count for example (see, for instance, [8]):

- (i) the number of ways of subdividing a convex polygon, with $n(k-1) + 2$ vertices, into n disjoint $k+1$ -gons by means of nonintersecting diagonals,
- (ii) the number of sequences $(a_1, a_2, \dots, a_{nk})$, where $a_i \in \{1, 1-k\}$, with all partial sums $a_1 + \dots + a_k$ nonnegative and with $a_1 + \dots + a_{nk} = 0$,

^{*}Department of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada, 89154-4020, USA.

[†]Department of Mathematics, Illinois Wesleyan University, Bloomington, Illinois 61702, USA

[‡]Department of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada, 89154-4020, USA. This work was completed while on sabbatical leave from University of Nevada, Las Vegas, and the author would like to thank UNLV for its support.

(iii) the number of noncrossing partitions π of $1, 2, \dots, n(k-1)$, such that $k-1$ divides the cardinality of every block of π ,

(iv) the number of k -cacti formed of n polygons, etc. See [1, 3, 4, 6, 16, 18, 19] for more details and examples.

The generating function $R_k(t)$ for the Fuss-Catalan numbers, $\{R_k(n, 1)\}_{n \geq 0}$ is called the generalized binomial series in [6], and it satisfies the function equation $R_k(t) = 1 + tR_k(t)^k$. Hence, from the Lambert's formula for the Taylor expansion of the powers of $R_k(t)$ (see [6]), we have

$$R_k^r \equiv R_k(t)^r = \sum_{n \geq 0} \frac{r}{mn+r} \binom{kn+r}{n} t^n \quad (2)$$

for all $r \in \mathbb{Z}$. Equation (2) implies the following formula of $R_k(t)$:

$$R_k(t) = 1 + tR_k^k(t). \quad (3)$$

Lobb [12] defines his Lobb numbers as

$$L_{n,m} := \frac{2n+1}{m+n+1} \binom{2n}{m+n}$$

for $n \geq m \geq 0$, which have the following combinatorial interpretation: Let $L_{n,m}$ be the number of sequences of length $2n$ with $n+m$ of the terms equal 1 and $n-m$ of the terms equal -1 . It is natural to extend Lobb numbers to the number of sequences with $(k-1)n+m$ terms equal to 1 and $n-m$ terms equal to $1-k$. We denote the extended Lobb numbers by $L_{n,m}^k$ and define them as

$$L_{n,m}^k := \frac{km+1}{(k-1)n+m+1} \binom{kn}{n-m}. \quad (4)$$

Generalized Lobb numbers include many number sequences as their special cases. For instance, when $k=2$, $L_{n,m}^2$ are classical Lobb numbers; when $m=0$,

$$L_{n,0}^k = \frac{1}{(k-1)n+1} \binom{kn}{n} =: C_k(n) \quad (5)$$

are the generalized Catalan numbers; when $k=2$ and $m=0$, then

$$L_{n,0}^2 = \frac{1}{n+1} \binom{2n}{n} =: C_2(n) \equiv C(n) \quad (6)$$

are the classical Catalan numbers; when $k=1$, then

$$L_{n,m}^1 = \binom{n}{m}$$

are the binomial numbers. Other special cases can be seen in [7, 8]. The following relationship between generalized Lobb numbers and Raney numbers make us switch our results between the generalized Lobb numbers and the Raney numbers (see, for example, [9]):

$$L_{n,m}^k = R_k(n-m, km+1), \quad (7)$$

which can be proved below. From (1) and using the transformation $n \rightarrow n - m$ and $r \rightarrow km + 1$, we have

$$\begin{aligned} R_k(n - m, km + 1) &= \frac{km + 1}{k(n - m) + km + 1} \binom{k(n - m) + km + 1}{n - m} \\ &= \frac{km + 1}{kn + 1} \binom{kn + 1}{n - m} = \frac{km + 1}{kn + 1} \frac{(kn + 1)!}{((k - 1)n + m + 1)!(n - m)!} \\ &= \frac{km + 1}{(k - 1)n + m + 1} \binom{kn}{n - m} = L_{n,m}^k, \end{aligned}$$

or equivalently,

$$L_{n+\frac{r-1}{k}, \frac{r-1}{k}}^k = R_k(n, r). \quad (8)$$

This paper is arranged as follows. In next section, we discuss the relationship between the generalized Lobb numbers and Raney numbers and the relationship between the generalized Lobb numbers and Ballot numbers. Some properties and identities of the generalized Lobb numbers are given. In Section 3, we discuss the divisibilities of the generalized Lobb numbers, Raney numbers, and generalized Catalan numbers.

2 Properties of the generalized Lobb numbers and Raney numbers

Proposition 2.1 *Let $L_{n,m}^k$ be defined by (4). Then*

$$L_{n,m}^k = \binom{kn}{n - m} - (k - 1) \binom{kn}{n - m - 1}. \quad (9)$$

Particularly,

$$L_{n,m}^2 = \frac{2m + 1}{n + m + 1} \binom{2n}{n - m} = \binom{2n}{n - m} - \binom{2n}{n - m - 1}. \quad (10)$$

For generalized Catalan numbers and Catalan numbers, there are

$$\begin{aligned} L_{n,0}^k &= C_k(n) = \binom{kn}{n} - (k - 1) \binom{kn}{n - 1} \quad \text{and} \\ L_{n,o}^2 &= C_2(n) = \binom{2n}{n} - \binom{2n}{n - 1}. \end{aligned} \quad (11)$$

Formula (9) also shows

$$L_{n,m}^1 = \binom{n}{m}.$$

Proof. The right-hand side of (9) generates

$$\begin{aligned} RHS &= \binom{kn}{n-m} - \frac{(k-1)(n-m)}{kn-n+m+1} \binom{kn}{n-m} \\ &= \left[1 - \frac{(k-1)(n-m)}{kn-n+m+1} \right] \binom{kn}{n-m} \\ &= \frac{km+1}{(k-1)n+m+1} \binom{kn}{n-m} = L_{n,m}^k. \end{aligned}$$

The results for special cases are straightforward from (9).

Proposition 2.2 Let $L_{n,m}^k$ be defined by (4). Then it can be written as

$$L_{n,m}^k = \frac{km+1}{kn+1} \binom{kn+1}{n-m}. \quad (12)$$

Particularly,

$$L_{n,m}^2 = \frac{2m+1}{2n+1} \binom{2n+1}{n-m} = \frac{2m+1}{n+m+1} \binom{2n}{n-m} = \frac{2m+1}{n+m+1} \binom{2n}{n+m}. \quad (13)$$

Proof. The right-hand side of (12) can be changed to

$$\begin{aligned} RHS &= \frac{km+1}{kn+1} \frac{(kn+1)!}{(n-m)!(kn-n+m+1)!} \\ &= \frac{km+1}{(k-1)n+m+1} \frac{(kn)!}{(n-m)!((k-1)n+m)!} = L_{n,m}^k. \end{aligned}$$

The special case (13) follows from (12).

Proposition 2.3 Let $L_{n,m}^k$ be defined by (4). Then

$$L_{n-m, \frac{r-1}{k}}^k = L_{n-m, \frac{r-2}{k}}^k + L_{n-m-1, \frac{r-2}{k}+1}^k. \quad (14)$$

Proof. From Corollary 3 of [14], we have

$$R_k(n, r) = R_k(n, r-1) + R_k(n-1, r+k-1), \quad (15)$$

which implies (14) by using (8).

Lobb numbers $L_{n,m}^2$ are also related to Ballot numbers (see, for example, [6])

$$B(a, b) = \frac{a-b}{a+b} \binom{a+b}{a} = \frac{a-b}{a+b} \binom{a+b}{b}. \quad (16)$$

Proposition 2.4 Let $L_{n,m}^k$ and $B(a, b)$ be defined by (4) and (16), respectively. Then

$$L_{n,m}^2 = B(n+m+1, n-m), \quad (17)$$

or equivalently,

$$B(n, m) = L_{\frac{n+m-1}{2}, \frac{n-m-1}{2}}^2. \quad (18)$$

Hence, $L_{n,m}^2$ is a special case of Ballot numbers.

Proof. Substituting $a = n + m + 1$ and $b = n - m$ yields

$$B(n + m + 1, n - m) = \frac{2m + 1}{2n + 1} \binom{2n + 1}{n - m} = L_{n,m}^2,$$

where the last equation is from (12).

Corollary 2.5 *Let $L_{n,m}^k$ be defined by (4). Then*

$$L_{n,m}^2 = L_{\frac{2n-1}{2}, \frac{2m-1}{2}}^2 + L_{\frac{2n-1}{2}, \frac{2m+1}{2}}^2. \quad (19)$$

Proof. From [6], we have

$$B(n, k) = B(n - 1, k) + B(n, k - 1).$$

Thus,

$$B(n + m + 1, n - m) = B(n + m, n - m) + B(n + m + 1, n - m - 1),$$

which implies (19) by using (18).

3 Divisibility of generalized Catalan numbers, generalized Lobb numbers, and Raney numbers

We now consider the divisibility properties of generalized Lobb numbers, generalized Catalan numbers, and Raney numbers.

Theorem 3.1 *Let $C_k(n) := \binom{kn}{n} / ((k - 1)n + 1)$ ($k \geq 2$), and let $n = (k + 1)t + 1$ ($t = 0, 1, 2, \dots$). Then*

(a) *If k is odd, then*

$$((k - 1)t + 1) \mid C_k(n). \quad (20)$$

(b) *If k is even and t is even, then*

$$((k - 1)t + 1) \mid C_k(n). \quad (21)$$

(c) *If k is even and t is odd, then*

$$((k - 1)t + 1) \mid 2C_k(n). \quad (22)$$

Proof. First, we express Lobb numbers $L_{n,m}^k$ in terms of generalized Catalan numbers $C_k(n)$:

$$\begin{aligned} L_{n,m}^k &= \frac{km + 1}{(k - 1)n + m + 1} \binom{kn}{n - m} \\ &= (km + 1) \frac{(kn)!}{(n - m)!((k - 1)n + m + 1)!} \\ &= (km + 1) \frac{(kn)!}{n!((k - 1)n + 1)!} \prod_{j=1}^m \frac{n - j + 1}{(k - 1)n + j + 1} \\ &= (km + 1) C_k(n) \prod_{j=1}^m \frac{n - j + 1}{(k - 1)n + j + 1}. \end{aligned}$$

Therefore, for non-negative integer t

$$\begin{aligned} L_{(k+1)t+1,1}^k &= C_k((k+1)t+1) \frac{(k+1)((k+1)t+1)}{(k-1)((k+1)t+1)+2} \\ &= C_k((k+1)t+1) \frac{(k+1)t+1}{(k-1)t+1}. \end{aligned} \quad (23)$$

Secondly, we consider different cases for k . In case (a), let k be odd, i.e., $k = 2l + 1$, $l = 0, 1, 2, \dots$. Then,

$$(k-1)t+1 = 2lt+1$$

and

$$(k+1)t+1 = 2(l+1)t+1.$$

Noting $(k+1)t+1 = (k-1)t+1+2t$, we have

$$\gcd[(k+1)t+1, (k-1)t+1] = \gcd[2t, (k-1)t+1] = 1$$

because $(k-1)t+1$ is an odd integer. From (23), we have proved $((k-1)t+1) \mid C_k((k+1)t+1)$ when k is odd. In case (b), we assume $k = 2l$ ($l \in \mathbb{Z}$), an even number. Then

$$\begin{aligned} (k-1)t+1 &= (2l-1)t+1, \\ (k+1)t+1 &= (2l+1)t+1 = (2l-1)t+1+2t. \end{aligned}$$

Thus,

$$\gcd[(k+1)t+1, (k-1)t+1] = \gcd[2t, (2l-1)t+1].$$

If t is even, then $\gcd[2t, (2l-1)t+1] = 1$, which implies $((k-1)t+1) \mid C_k((k+1)t+1)$. Finally, considering the case (c), where k is an even number $2l$ and t is an odd number $2u+1$ ($l, u \in \mathbb{Z}$), we have

$$\begin{aligned} \gcd[2t, (2l-1)t+1] &= \gcd[2(2u+1), (2l-1)(2u+1)+1] \\ &= \gcd[2(2u+1), -2u] = 2 \end{aligned}$$

So that $((k-1)t+1) \mid 2C_k((k+1)t+1)$, which completes the proof.

Example 3.1 For $k = 3$ and $t = 1$, we have $(k-1)t+1 = 3$ and $(k+1)t+1 = 5$. $C_3(5) = 273$ and $3 \mid C_3(5)$. For $k = 3$ and $t = 2$, we have $(k-1)t+1 = 5$ and $(k+1)t+1 = 9$. Thus $5 \mid C_3(9) = 246675$. For $k = 2$ and $t = 2$, we have $(k-1)t+1 = 3$ and $(k+1)t+1 = 7$, which implies $3 \mid C_2(7) = 429$.

Example 3.2 For $k = 3$, from Theorem 3.1 there holds $2t+1 \mid C_3(4t+1)$. Thus,

$$1 \mid C_3(1), 3 \mid C_3(5), 5 \mid C_3(9), 7 \mid C_3(13), 9 \mid C_3(17), \text{etc.}$$

Here, $\{2t+1 : t = 0, 1, 2, \dots\}$ and $\{4t+1 : t = 0, 1, 2, \dots\}$ form arithmetical sequences.

Remark From the expression of Lobb numbers $L_{n,m}^k$ in terms of generalized Catalan numbers $C_k(n)$, we have

$$L_{n,1}^k = \frac{k+1}{(k-1)n+2} \binom{kn}{n-1} = (k+1)C_k(n) \frac{n}{(k-1)n+2}. \quad (24)$$

Hence, provided

$$\gcd((k+1)n, (k-1)n+2) = 1,$$

or equivalently,

$$\gcd((k+1)n, -2(n-1)) = 1, \quad (25)$$

we have

$$((k-1)n+2) | C_k(n). \quad (26)$$

Note (25) implies that $(k+1)n$ must be odd, or equivalently, n is odd and k is even. In other words, if k is even and t is even, $C_k((k+1)t+1)$ has two divisors $(k-1)t+1$ and $(k^2-1)t+k+1$, which are given by (21) and (26) respectively.

Example 3.3 If $n = 3$ and $k = 2$, then $\gcd((k+1)n, (k-1)n+2) = \gcd(9, 4) = 1$. From (26), $5 | C_2(3)$. Actually, $C_2(3) = 5$. Similarly, if $n = 3$ and $k = 4$, then $\gcd(11, 4) = 1$, which implies $11 | C_4(3)$. Actually, $C_4(3) = 22$. While $n = 3$ and $k = 6$ yield $17 | C_6(3)$, where $C_6(3) = 51$, and $n = 5$ and $k = 2$ yield $7 | C_2(5)$, where $C_2(5) = 42$. An non-example is given by $n = 7$ and $k = 2$, which yields $\gcd(21, -12) = 3 \neq 1$. Since $C_2(7) = 429$, which does not have divisor 21.

Sury [20] proves if $n \neq (p^l - 1)/(p - 1)$ for any prime $p \geq 3$, then

$$p | C_p(n). \quad (27)$$

A natural question is what is a divisor of $C_p((p^l - 1)/(p - 1))$. We now apply Theorem 3.1 to answer this question.

Corollary 3.2 Let $C_k(n)$ be the generalized Catalan numbers defined by (5). Then for an odd integer l we have

$$\frac{p^l + 1}{p + 1} \Big| C_p \left(\frac{p^l - 1}{p - 1} \right) \quad (p \geq 3). \quad (28)$$

Proof. If $k = p \geq 3$ and $n = (k+1)t+1 = (p^l - 1)/(p - 1)$, then

$$t = \frac{1}{k+1} \left(\frac{p^l - 1}{p - 1} - 1 \right) = \frac{p^l - p}{p^2 - 1},$$

where t is an integer because l is odd. Thus

$$(k-1)t+1 = (k+1)t+1 - 2t = \frac{p^l - 1}{p - 1} - 2 \frac{p^l - p}{p^2 - 1} = \frac{p^l + 1}{p + 1}.$$

Substituting $k = p$, $n = (k+1)t+1 = (p^l - 1)/(p - 1)$, and $(k-1)t+1 = (p^l + 1)/(p + 1)$ into (20) of Theorem 3.1, we obtain (28).

In [11, 10], the following result is given

$$(2^{l+1} - 3) \mid C_2(M_l),$$

where M_l are the Mersenne numbers, $2^l - 1$ ($l = 0, 1, 2, \dots$), and $C_2(n) = L_{n,0}^2$ are classical Catalan numbers. Thus, for $l = 4$ and 5 , there are $29 \mid C_2(M_4)$ and $61 \mid C_2(M_5)$, respectively.

We obtain the following corollary from Theorem 3.1, which extends the results shown in [11, 10].

Corollary 3.3 *Let $C_2(n)$ be the Catalan numbers. Then*

$$\frac{2^l + 1}{3} \mid C_2(2^l - 1) \quad (29)$$

for $l = 1, 3, 5, 7, \dots$. Combining [11], $C_2(M_k)$ has two different divisors, $2^{l+1} - 3$ and $(2^l + 1)/3$, when odd $l > 1$. Furthermore, if l is odd and not a prime, then all of its divisors are divisors of $C_2(2^l - 1)$.

Proof. Set $(k + 1)t + 1 = 2^l - 1$. Then $t = (2^l - 2)/(k + 1)$. Let $k = 2$, we have $t = (2^l - 2)/3$. Here t is even because

$$3t = 2^l - 2$$

is even. Thus,

$$(k - 1)t + 1 = (2 - 1)t + 1 = \frac{2^l + 1}{3}.$$

From Theorem 3.1 (b), for $k = 2$ and $t = (2^l - 2)/3$ we obtain

$$\frac{2^l + 1}{3} = ((k - 1)t + 1) \mid C_k((k + 1)t + 1) = C_2(2^l - 1)$$

for $l = 1, 3, 5, 7, \dots$. To prove that $C_2(M_l)$ has two different divisors, $2^{l+1} - 3$ and $(2^l + 1)/3$, when odd $l > 1$, we only need to show

$$2^{l+1} - 3 \neq \frac{2^l + 1}{3}$$

when $l > 1$. This is clearly true, otherwise, there is a contradiction

$$3 \cdot 2^l - 2^{l-1} = 5$$

for $l > 1$. Finally, from [2], we know that $(2^l + 1)/3$ is a prime only if l is a prime. Hence, if l is not a prime number, then $(2^l + 1)/3$ is a composite number. Additionally, when l is odd and not a prime, then all of the divisors of such composite number are also divisors of $C_2(2^l - 1)$ because of (29).

Example 3.4 For $l = 1, 3, 5$, and 7 , Corollary 3.3 generates $\frac{2^l + 1}{3} \mid C_2(2^l - 1)$ for $l = 1, 3, 5$, and 7 . For examples,

$$1 \mid C_2(1), 3 \mid C_2(7), 11 \mid C_2(31), \text{ and } 43 \mid C_2(127).$$

Among the above results, the second and fourth are new. Actually, we may give infinitely many new results from Corollary 3.3.

We now extend the result on Catalan numbers shown in Corollary 3.3 to generalized Catalan numbers.

Theorem 3.4 *Let $C_k(n) := L_{n,0}^k$ be the generalized Catalan numbers defined by (5). If k is even and $\ell \equiv 1 \pmod{\phi(k+1)}$, where $\phi(n)$ is Euler's totient function, then*

$$\left((k-1) \frac{2^l - 2}{k+1} + 1 \right) \Big| C_k(2^l - 1). \quad (30)$$

Proof. Let

$$(k+1)t + 1 = 2^l - 1,$$

the Mersenne numbers. Then $t = (2^l - 2)/(k+1)$, where t is even because k is even, and

$$(k-1)t + 1 = (k-1) \frac{2^l - 2}{k+1} + 1.$$

To prove (30), we need to show the right-hand side of the above equation is an integer, i.e.,

$$(k-1)(2^l - 2) \equiv 0 \pmod{k+1}.$$

The last equation is equivalent to

$$-4(2^{l-1} - 1) \equiv 0 \pmod{k+1}$$

because

$$(k-1)(2^l - 2) = (k+1)(2^l - 2) - 4(2^{l-1} - 1).$$

Therefore, if $\gcd(4, k+1) = 1$, then we need

$$2^{l-1} \equiv 1 \pmod{k+1}. \quad (31)$$

From Euler theorem, if $\gcd(2, k+1) = 1$; i.e., k is even, then

$$2^{\phi(k+1)} \equiv 1 \pmod{k+1},$$

where $\phi(n)$ is Euler's totient function, i.e., the number of the positive integers less than or equal to n that are relatively prime to n . Comparing the above equation and equation (31), we should have

$$l - 1 \equiv 0 \pmod{\phi(k+1)},$$

or equivalently,

$$\ell = u\phi(k+1) + 1$$

for some integer u . Now, we assume that k is even and $\ell \equiv 1 \pmod{\phi(k+1)}$, where ϕ is Euler's totient function. Under the conditions, $((k-1)t + 1 = (k-1)(2^l - 2)/(k+1))$ is an integer when $t = (2^l - 2)/(k+1)$. Now k is even and t is even. Then by Theorem 3.1 (b)

$$\left((k-1) \frac{2^l - 2}{k+1} + 1 \right) \Big| C_k(2^l - 1).$$

Example 3.4 Let $C_k(n) := L_{n,0}^k$ be the generalized Catalan numbers defined by (5). Since $k = 4$ is even, and $\phi(k+1) = \phi(5) = 4$, from Theorem 3.4, for $l \equiv 1 \pmod{4}$, i.e., $l = 1, 5, 9, \dots$, we have

$$\left((4-1) \frac{2^l - 2}{4+1} + 1 \right) \Big| C_4(2^l - 1),$$

which implies

$$\frac{3 \cdot 2^l - 1}{5} \Big| C_4(2^l - 1) \quad (32)$$

for $l = 1, 5, 9, \dots$

In Theorem 3.4, the condition $l \equiv 1 \pmod{\phi(k+1)}$ can be replaced by $l \equiv 1 \pmod{k}$ when $k+1$ is a prime number greater than 3. In this case, the condition of that k is even is automatically satisfied. Hence, we have the following corollary of Theorem 3.4.

Corollary 3.5 Let $C_k(n) := L_{n,0}^k$ be the generalized Catalan numbers defined by (5). If $k+1$ is a prime number greater than 3, and $l \equiv 1 \pmod{k}$, then

$$\left((k-1) \frac{2^l - 2}{k+1} + 1 \right) \Big| C_k(2^l - 1). \quad (33)$$

Proof. It is sufficient to note that if $k+1$ is a prime number greater than 3, then k is an even number and $\phi(k+1) = k$. Hence, Theorem 3.4 implies the corollary.

From the above discussion, the key to get divisibility of $C_k(n)$ by using (23) is

$$((k-1)t+1) \Big| (k+1)t+1.$$

Hence, we may have a special case of Theorem 3.1, which is more easier to be applied.

Example 3.6 Let $C_k(n) := L_{n,0}^k$ be the generalized Catalan numbers defined by (5). If t is even, then

$$(t+1) \Big| C_2(3t+1) \quad \text{and} \quad (3t+1) \Big| C_4(5t+1). \quad (34)$$

Thus,

$$1 \Big| C_2(1), 3 \Big| C_2(7), 5 \Big| C_2(13), 7 \Big| C_2(19), 9 \Big| C_2(25), \text{etc.}$$

and

$$1 \Big| C_4(1), 7 \Big| C_4(11), 13 \Big| C_4(21), 19 \Big| C_4(31), 25 \Big| C_4(41), \text{etc.}$$

In general, if $t = 2m$, then we have

$$(2m+1) \Big| C_2(6m+1), (6m+1) \Big| C_4(10m+1), (10m+1) \Big| C_6(14m+1), \text{etc.}$$

for $k = 2, 4, 6$, etc. More generally, for $k = 2u$ and $t = 2m$, we have

$$(2(2u-1)m+1) \Big| C_{2u}(2(2u+1)m+1),$$

where the sequences of $\{2(2u-1)m+1t = 0, 1, 2, \dots\}$, $\{2(2u-1)m+1m = 0, 1, 2, \dots\}$, $\{2(2u+1)m+1t = 0, 1, 2, \dots\}$, and $\{2(2u+1)m+1m = 0, 1, 2, \dots\}$ are arithmetical sequences.

We now transfer the divisibility from the generalized Lobb numbers to Raney numbers and Ballot numbers.

Theorem 3.6 *Let $R_k(n, m)$ be Raney numbers defined by (1). If k is an odd integer, then we have*

$$((k-1)t+1) \mid R_k((k+1)t+1, 1). \quad (35)$$

If k is an even integer and $n = (k+1)t+1$ is odd, then (35) holds. If both k and $n = (k+1)t+1$ are even, then

$$((k-1)t+1) \mid 2R_k((k+1)t+1, 1) \quad (36)$$

holds.

Proof. By using the relationship (7) between the generalized Lobb numbers and Raney numbers, we may establish Theorem 3.6 from Theorem 3.1.

Theorem 3.7 *Let $B(a, b)$ be Ballot numbers defined by (16). If $n = 3t+1$ is odd, then we have*

$$((k-1)t+1) \mid B((k+1)t+2, (k+1)t+1). \quad (37)$$

If $n = 3t+1$ is even, then t is odd and

$$((k-1)t+1) \mid 2B((k+1)t+2, (k+1)t+1) \quad (38)$$

holds.

Proof. From the relationship between $L_{n,m}^2$ and $B(a, b)$ shown in (17) and Theorem 3.1, we may obtain (37) and (38).

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