# Riordan Arrays Associated with Laurent Series and Generalized Sheffer-Type Groups

Tian-Xiao He[1]

[1]Department of Mathematics and Computer Science

Illinois Wesleyan University

Bloomington, IL 61702-2900, USA

the@iwu.edu

February 11, 2011

**Abstract**

A relationship between a pair of Laurent series and Riordan arrays is formulated. In addition, a type of generalized Sheffer groups is defined using Riordan arrays with respect to power series with non-zero coefficients. The isomorphism between a generalized Sheffer group and the group of the Riordan arrays associated with Laurent series is established. Furthermore, Appell, associated, Bell, and hitting-time subgroups of the groups are defined and discussed. A relationship between the generalized Sheffer groups with respect to different power series is presented. The equivalence of the defined Riordan array pairs and generalized Stirling number pairs is given. A type of inverse relations of various series is constructed using pairs of Riordan arrays. Finally, several applications involving various arrays, polynomial sequences, special formulas and identities are also presented as illustrative examples.

**Key words** Riordan arrays, Laurent series, Sheffer-type polynomial sequence, generalized Sheffer group, generalized Stirling numbers, Appell subgroup, associated subgroup, Bell subgroup, and hitting-time subgroup.

**Mathematics subject Classification (2000)** 05A30, 05A15, 05A10, 05A19

# 1 Introduction

In the recent literature, special emphasis has been given to the concept of *Riordan arrays* associated with power series, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [18]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli [20, 21], on subgroups of the Riordan group in Peart and Woan [12] and Shapiro [15], on some characterizations of Riordan matrices in Rogers [14], Merlini et al. [10], and He et al. [7], and on many interesting related results in Cheon et al. [2, 3], He et al. [6], Nkwanta [11], Shapiro [16, 17], and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{C}[\![z]\!]$; the *order* of $f(z) \in \mathcal{F}$, $f(z) = \sum_{k=0}^{\infty} f_k z^k$ ($f_k \in \mathbb{C}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order $r$. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s. $f(z)$ for which the compositional inverse $\overline{f}(z)$ exists such that $f(\overline{f}(z)) = \overline{f}(f(z)) = z$. Let $d(z) \in \mathcal{F}_0$ and $h(z) \in \mathcal{F}_1$; the pair $(d(z), h(z))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}}$ having

$$d_{n,k} = [z^n]d(z)h(z)^k \tag{1}$$

or, in other words, having $d(z)h(z)^k$ as the generating function whose coefficients make-up the entries of column $k$. Rogers [14] introduced the concept of the *A-sequence* for Riordan arrays; Merlini et al. [10] introduced the related concept of the *Z-sequence* and showed that these two concepts, together with the element $d_{0,0}$, completely characterize

The concept of a Riordan array can be extended in various ways, as shown in Corsani et al [4] so that the term "proper" distinguishes the arrays belonging to the Riordan group. In the present paper we will be interested only in this latter kind of Riordan arrays, so we usually understand the qualification *proper* and use it only to stress that some property holds for proper, but may not hold for non-proper Riordan arrays.

a proper Riordan array. In [7], Sprugnoli and the author consider the characterization of Riordan arrays by means of the $A$- and $Z$-sequences. [7] also shows how the $A$- and $Z$-sequences of the product of two Riordan arrays are derived from those of the two factors; similar results are obtained for the inverse. How the sequence characterization is applied to construct easily a Riordan array is presented in the paper. Finally, it gives the characterizations relative to some subgroups of the Riordan group, in particular of the hitting-time subgroup.

It is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(z),\, h_1(z)) * (d_2(z),\, h_2(z)) = (d_1(z)d_2(h_1(z)),\, h_2(h_1(z))). \tag{2}$$

The Riordan array $I = (1,\, z)$ is everywhere 0 except that it contains all 1's on the main diagonal; it is easily seen that $I$ acts as an identity for this product, that is, $(1,\, z) * (d(z),\, h(z)) = (d(z),\, h(z)) * (1, z) = (d(z),\, h(z))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(z),\, h(z))^{-1} = \left(\frac{1}{d(\overline{h}(z))},\, \overline{h}(z)\right) \tag{3}$$

where $\overline{h}(z)$ is the compositional inverse of $h(z)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays is a group.

If $(f_k)_{k\in\mathbb{N}}$ is any sequence and $f(z) = \sum_{k=0}^{\infty} f_k z^k$ is its generating function, then for every Riordan array $D = (d(z),\, h(z))$ we have:

$$\sum_{k=0}^{n} d_{n,k} f_k = [z^n] d(z) f(h(z))$$

which relates Riordan arrays to combinatorial sums and sum inversion.

Let $\Sigma$ be the set of functions with a Laurent expansion of the form

$$f(z) = \sum_{n\geq -1} a_n z^{-n}, \tag{4}$$

$a_n \in \mathbb{C}$. The order of $f(z) \in \Sigma$ is the maximum number $r \in \mathbb{Z}$ such that $f_r \neq 0$. $\Sigma_r$ is the set of formal power series of order $r$. It is known that $f \in \Sigma_0$ implies $1/f(z^{-1}) \in \mathcal{F}_0$ the set of invertible f.p.s, and $f \in \Sigma_{-1}$ implies $1/f(z^{-1}) \in \mathcal{F}_1$, the set of compositionally invertible

f.p.s. If $f(z) \in \Sigma_0$ and $g(z) \in \Sigma_{-1}$, we now define the Riordan arrays associated with Laurent series as follows. Similar to (1), we have matrix with entries

$$d_{n,k} = [z^n] \frac{1}{f(z^{-1})} \left( \frac{1}{g(z^{-1})} \right)^k, \tag{5}$$

which is called the Riordan array associated with $f(z)$ and $g(z)$ and denote it by $(1/f(z^{-1}),\, 1/g(z^{-1}))$. Hence, the generating function of column k of the matrix is $1/(f(z^{-1})g(z^{-1})^k)$. There exists a relationship between the Riordan arrays associated with power series and the Riordan arrays associated with Laurent series by using the transformation

$$T : f(z) \mapsto d(z), \quad d(z) = (Tf)(z) = \frac{1}{f(z^{-1})}. \tag{6}$$

It is obvious $T$ is well defined in the sense that if $f$ is in respectively $\Sigma_0$ and $\Sigma_{-1}$, then $d \in \mathcal{F}_0$ and $d \in \mathcal{F}_1$ exists uniquely. Denote by $\bar{\mathcal{R}}$ the set of all Riordan arrays associated with Laurent series, $[f(z),\, g(z)]$, where $f \in \Sigma_0$ and $g \in \Sigma_{-1}$. We now define $[f(z),\, g(z)]$ by using the Riordan arrays associated with power series as follows.

$$[f(z),\, g(z)] := \left( \frac{1}{f\,(z^{-1})},\, \frac{1}{g\,(z^{-1})} \right) \equiv (d(z),\, h(z)), \tag{7}$$

where $f \in \Sigma_0$ and $g \in \Sigma_{-1}$, which implies $d(z) = 1/f(z^{-1}) \in \mathcal{F}_0$ and $h(z) = 1/g(z^{-1}) \in \mathcal{F}_1$, or equivalently, $(d(z),\, h(z)) \in \mathcal{R}$.

Denote by $\mathcal{E}$ the set of all power series $\sum_{n\geq 0} c_n z^n$ with all $c_n \neq 0$. In [5], we defined a generalized Sheffer-type polynomial sequences using the expansion

$$d(z) A(xh(z)) = \sum_{n\geq 0} p_n(x) z^n, \tag{8}$$

where $d(z) \in \mathcal{F}_0$, $h(z) \in \mathcal{F}_1$, and $A(z) = \sum_{n\geq 0} a_n z^n \in \mathcal{E}$. We now re-state the definition associated with Laurent series. Unless otherwise specified, when dealing with $A(z)$ we shall assume it is in $\mathcal{E}$.

**Definition 1.1** *Let $f \in \Sigma_0$ and $g \in \Sigma_{-1}$. A generalized Sheffer-type polynomial sequence associated with Laurent series generated by $[f,\, g]$ with respect to a power series $A(z) = \sum_{n\geq 0} a_n z^n$ is defined by*

$$\frac{1}{f(z^{-1})} A\left( \frac{x}{g(z^{-1})} \right) = \sum_{n\geq 0} p_n(x) z^n, \tag{9}$$

*or equivalently,*

$$\frac{1}{f(z)}A\left(\frac{x}{g(z)}\right)=\sum_{n\geq 0}p_n(x)z^{-n}. \tag{10}$$

*Furthermore, if* $a_n = 1$ *(*$n = 0, 1, 2, \ldots$*), i.e.,* $A(z) = 1/(1 - z)$*, then the corresponding sequences* $p_n(x)$ *defined by (9) are called the ordinary polynomial sequences. If* $a_n = 1/n!$ *(*$n = 0, 1, 2, \ldots$*), i.e.,* $A(z) = e^z$*, then expression (9) defines the classic Sheffer-type polynomial sequences. If* $a_0 = 1$ *and* $a_n = 1/n$ *(*$n = 1, 2, \ldots$*), i.e.,* $A(z) = 1-\ln(1-z)$*, then the corresponding* $p_n(x)$ *are called the Dirichlet polynomial sequences.*

Denote by $P_A$ the set of all generalized Sheffer-type polynomial sequences $\{p_n(x)\}$ generated by (9) or an equivalent form (8).

By using the Riordan array defined above and the fundamental theorem for Riordan arrays (see [15]), we have the expression of the generalized Sheffer-type polynomial sequences defined by (9)

$$p_n(x)=\sum_{k=0}^{n}a_k d_{n,k}x^k=\sum_{k=0}^{n}p_{n,k}x^k, \tag{11}$$

where $p_{n,k} = a_n d_{n,k}$ and

$$d_{n,k}=[z^n]\frac{1}{f(z^{-1})}\left(\frac{1}{g(z^{-1})}\right)^k, \tag{12}$$

which can be obtained from (9) with an observation of (11), the expression of $p_n(x)$. Hence, the matrix form of $\frac{1}{f(z^{-1})}A\left(\frac{x}{g(z^{-1})}\right)$ is the result of the following matrix multiplication:

$$\left(\frac{1}{f(z^{-1})},\frac{1}{f(z^{-1})}\left(\frac{1}{g(z^{-1})}\right),\frac{1}{f(z^{-1})}\left(\frac{1}{g(z^{-1})}\right)^2,\ldots\right)(a_0,a_1x,a_2x^2,\ldots)^T.$$

For the sake of symmetry, one may write the generalized Sheffer-type polynomial sequences defined by (8) as $\{\tilde{p}_n(x)\}$, which is defined by

$$d(z)A(xh(z))=\sum_{n\geq 0}a_n\tilde{p}_n(x)z^n, \tag{13}$$

where

$$a_n\tilde{p}_n(x) = p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k = \sum_{k=0}^{n} a_k d_{n,k}x^k.$$

If $a_n \neq 0$ for $n \in \mathbb{N}$, then the coefficient matrix of polynomial sequence $\{\tilde{p}_n(x)\}$ is

$$\left(\frac{p_{n,k}}{a_n}\right)_{n\geq k\geq 0} = \left(\frac{a_k}{a_n}d_{n,k}\right)_{n\geq k\geq 0} = D^{-1}(d_{n,k})_{n\geq k\geq 0}D,$$

where $D = diag(a_0, a_1, a_2, \ldots)$. The last matrix is called a (c)-Riordan array, where $c = (a_0, a_1, a_2, \ldots)$, which is also called a generalized Riordan array in [22]. We denote

$$\sigma(n,k) := \frac{a_k}{a_n}d_{n,k} \tag{14}$$

and call it the *generalized Stirling number* associated with $(d(z),\ h(z))$ and $A(z)$. Hence, the generalized Stirling numbers are the entries of a generalized Riordan array. The generalized Stirling numbers can be considered as an extension of the weighted Stirling numbers. There are two special kinds of weighted Stirling numbers defined by Carlitz [4] (see also [1], [8], and [9]). The Stirling numbers of the first kind and second kind are the special cases of (14) with $d_{n,k}$ generated by $(d(z), h(z)) = (1, \log(1+z))$ and $(d(z), h(z)) = (1, e^z - 1)$, respectively (see, for example, [13]).

**Example 1.1** As an example of the generalized Sheffer-type polynomial sequence, suppose $f \in \Sigma_0$ and $g \in \Sigma_{-1}$ and denote $d(z) = 1/f(z^{-1})$ and $h(z) = 1/g(z^{-1})$. Thus $d(z) \in \mathcal{F}_0$ and $h(z) \in \mathcal{F}_1$. The ordinary Sheffer-type polynomial sequence $\{p_n(x)\}$ is generated by

$$\frac{d(z)}{1 - xh(z)} = \sum_{n\geq 0} p_n(x)z^n, \tag{15}$$

or equivalently,

$$\frac{g(z)}{f(z)(g(z) - x)} = \sum_{n\geq 0} p_n(x)z^{-n}, \tag{16}$$

where

$$p_n(x) = \sum_{k=0}^{n} d_{n,k}x^k = \sum_{k=0}^{n} [z^n]d(z)(h(z))^k x^k.$$

If $d(z) = \frac{zh'(z)}{h(z)}$, or equivalently, $f(z) = \frac{g(z)}{zg'(z)}$, then the corresponding $\{F_n(x)\}$ defined by

$$\frac{zh'(z)}{h(z)(1-xh(z))} = \sum_{n\geq 0} F_n(x)z^n$$

is the Faber polynomial sequence (see [3]). By substituting $h(z) = 1/g(z^{-1})$ and replacing $z$ by $z^{-1}$, we have

$$\frac{g'(z)}{g(z)-x} = \sum_{n\geq 0} \frac{F_n(x)}{z^{n+1}}$$

(see [19]).

In next section, we will show the operation # defined below is closed in $\bar{\mathcal{R}}$:

$$[f_1(z),\, g_1(z)]\#[f_2(z),\, g_2(z)] = [f_1(z)f_2(g_1(z)),\, g_2(g_1(z))], \tag{17}$$

and $(\bar{\mathcal{R}}, \#)$ forms a group. In addition, for any $\{p_n(x)\}$ and $\{q_n(x)\} \in P_A$, we will show the operation $\tilde{\#}$ defined by

$$\{p_n(x)\}\tilde{\#}\{q_n(x)\} = \{r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k : r_{n,k} = \sum_{\ell=k}^{n} p_{n,\ell}g_{\ell,k}/a_\ell, n \geq \ell \geq k\} \tag{18}$$

is closed in $P_A$, and $(P_A, \tilde{\#})$ forms a group. An isomorphism between $(\bar{\mathcal{R}}, \#)$ and $(P_A, \tilde{\#})$ will also be proved. Furthermore, the corresponding subgroups of $(\bar{\mathcal{R}}, \#)$ and $(P_A, \tilde{\#})$ that are isomorphic to the Appell, Lagrange, Bell, and hitting-time subgroups of classic Riordan group will be given in Section 3. Finally, we present the relationships between the generalized Sheffer-type polynomial sequences from different groups in $\{P_A : A \in \mathcal{E}\}$, and we will establish inverse relations for pairs of generalized Sheffer-type polynomial sequences in the same group of $P_A$. In addition, an inverse relation of power series using pairs of Riordan arrays and pairs of generalized Stirling numbers will be given.

## 2 Riordan group associated with Laurent series and the group of generalized Sheffer-type polynomial sequences

First, we prove that $\bar{\mathcal{R}}$ is closed under the operation defined in (17). Indeed,

$$\begin{aligned} & [f_1(z),\, g_1(z)]\#[f_2(z),\, g_2(z)] \\ = & \left(\frac{1}{f_1\left(z^{-1}\right)},\, \frac{1}{g_1\left(z^{-1}\right)}\right) * \left(\frac{1}{f_2\left(z^{-1}\right)},\, \frac{1}{g_2\left(z^{-1}\right)}\right) \\ = & \left(\frac{1}{f_1\left(z^{-1}\right)} \frac{1}{f_2\left(g_1\left(z^{-1}\right)\right)},\, \frac{1}{g_2\left(g_1\left(z^{-1}\right)\right)}\right) \\ = & [f_1(z) f_2(g_1(z)),\, g_2(g_1(z))]. \end{aligned} \tag{19}$$

It is easy to see that the operation # defined in (17) satisfies the associative law. In addition, for any $[f(z),\, g(z)] \in \bar{\mathcal{R}}$, we have

$$[f(z),\, g(z)]\#[1,\, z] = [f(z),\, g(z)],\ \textit{and}\ [f(z),\, g(z)]^{-1} = \left[\frac{1}{f(\bar{g}(z))},\, \bar{g}(z)\right], \tag{20}$$

where $\bar{g}(z^{-1})$ is the compositional inverse of $g(z^{-1})$ in terms of $z^{-1}$, i.e.,

$$\bar{g}(g(z^{-1})) = g(\bar{g}(z^{-1})) = z^{-1}.$$

The first equation of (20) can be proved from the definition of operation #:

$$[f(z),\, g(z)]\#[1,\, z] = \left(\frac{1}{f\left(z^{-1}\right)},\, \frac{1}{g\left(z^{-1}\right)}\right) * \left(1,\, \frac{1}{z^{-1}}\right) = [f(z),\, g(z)].$$

Similarly, by noting the inverse of $1/g(z^{-1})$ is $1/\bar{g}(z^{-1})$, we have

$$\begin{aligned} & [f(z),\, g(z)]\#\left[\frac{1}{f(\bar{g}(z))},\, \bar{g}(z)\right] \\ = & \left(\frac{1}{f\left(z^{-1}\right)},\, \frac{1}{g\left(z^{-1}\right)}\right) * \left(f\left(\bar{g}\left(z^{-1}\right)\right),\, \frac{1}{\bar{g}\left(z^{-1}\right)}\right) \\ = & \left(\frac{1}{f\left(z^{-1}\right)} f\left(\bar{g}\left(g\left(z^{-1}\right)\right)\right),\, \frac{1}{\bar{g}\left(g\left(z^{-1}\right)\right)}\right) = [1,\, z]. \end{aligned}$$

Thus, the second equation of (20) is obtained. Surveying the above results, we have

**Theorem 2.1** *The set $\bar{\mathcal{R}}$ forms a group under the operation # defined in (17).*

We call $\bar{\mathcal{R}}$ the Riordan group associated with Laurent series. Since the mapping defined in (6) from pair $(f(z),\, g(z))$, $f \in \Sigma_0$ and $g \in \Sigma_{-1}$, to pair $(d(z),\, h(z))$, $d \in \mathcal{F}_0$ and $h \in \mathcal{F}_1$, is one-to-one and onto, we immediately have the following result.

**Proposition 2.2** *There exists a one-to-one correspondence between groups $\bar{\mathcal{R}}$ and Riordan group $\mathcal{R}$.*

In the sense shown in Proposition 2.2, we may say that the usual Riordan group $\mathcal{R}$ is essentially same as the group $\bar{\mathcal{R}}$ of extended Riordan arrays. The only difference is whether it is associated to power series or Laurent series.

Similarly to Theorem 2.1, we can prove the following result.

**Theorem 2.3** *For a power series $A(z) = \sum_{n\geq 0} a_n z^n$, $(P_A, \tilde{\#})$ forms a group called the generalized Sheffer group associated with $A$.*

*Proof.* Suppose sequences $\{p_n(x)\},\ \{q_n(x)\} \in P_A$ are generated by $[f_1,\, g_1]$ and $[f_2,\, g_2]$, respectively. Then,

$$p_n(x) = \sum_{k=0}^{n} a_k [z^n] \frac{1}{f_1(z^{-1})} \left(\frac{1}{g_1(z^{-1})}\right)^k x^k = \sum_{k=0}^{n} a_k d_{n,k} x^k$$
$$q_n(x) = \sum_{k=0}^{n} a_k [z^n] \frac{1}{f_2(z^{-1})} \left(\frac{1}{g_2(z^{-1})}\right)^k x^k = \sum_{k=0}^{n} a_k c_{n,k} x^k.$$

From (18), the element of $\{r_n(x)\} = \{p_n(x)\}\tilde{\#}\{q_n(x)\}$ is

$$\begin{aligned} r_n(x) &= \sum_{k=0}^{n} \left( \sum_{\ell=k}^{n} a_k d_{n,k} c_{n,k} \right) x^k \\ &= \sum_{k=0}^{n} a_k [z^n] \frac{1}{f_1(z^{-1}) f_2(g_1(z^{-1}))} \left(\frac{1}{g_2(g_1(z^{-1}))}\right)^k x^k, \end{aligned} \tag{21}$$

i.e., $\{r_n(x)\} \in P_A$ is a generalized Sheffer-type polynomial sequence generated by $[f_1 f_2 \circ g_1,\, g_2 \circ g_1]$. Hence, $P_A$ is closed under the operation $\tilde{\#}$ defined by (18).

Since

$$\begin{aligned}\left(\{p_n(x)\}\tilde{\#}\{q_n(x)\}\right)\tilde{\#}\{r_n(x)\} &= \left\{\sum_{k=0}^{n}\left(\sum_{u=k}^{n}\sum_{\ell=u}^{n}\ell!u!p_{n,\ell}q_{\ell,u}r_{u,k}\right)x^k\right\}\\ &= \{p_n(x)\}\tilde{\#}\left(\{q_n(x)\}\tilde{\#}\{r_n(x)\}\right)\end{aligned}$$

for every $\{p_n(x) = \sum_{k\geq 0} p_{n,k}x^k\}$, $\{q_n(x) = \sum_{k\geq 0} q_{n,k}x^k\}$, and $\{r_n(x) = \sum_{k\geq 0} r_{n,k}x^k\}$ in $P_A$, the operation $\tilde{\#}$ satisfies the associative law. In addition, we have the multiplication identity of $P_A$ as $\{a_n x^n\}$ because of

$$\{p_n(x)\}\tilde{\#}\{a_n x^n\} = \{\sum_{n\geq 0}\left(\sum_{\ell=k}^{n} p_{n,\ell}a_\ell\delta_{\ell,k}/a_\ell\right)x^k = \sum_{n\geq 0}p_{n,k}x^k = p_n(x)\}$$

for every $\{p_n(x)\} \in P_A$.

Finally, it can be easily checked that the inverse of $\{p_n(x)\} \in P_A$ is $\{p_n(x)\}^{-1}$ generated by $f(\bar{g}(z^{-1}))A(x/\bar{g}(z^{-1}))$, where $\bar{g}(g(z^{-1})) = g(\bar{g}(z^{-1})) = z^{-1}$.

■

We now establish a relationship between groups $(\bar{R}, \#)$ and $(P_A, \tilde{\#})$.

**Theorem 2.4** *Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a power series. Then Riordan group $(\bar{R}, \#)$ associated with Laurent series is isomorphic to the generalized Sheffer group $(P_A, \tilde{\#})$ associated with $A$.*

*Proof.* For the power series $A(z) = \sum_{n\geq 0} a_n z^n$, we define a map

$$\theta_A : \bar{R} \mapsto P_A$$

by

$$\theta_A[f(z),\, g(z)] = \left\{[z^n]\frac{1}{f(z^{-1})}A\left(\frac{x}{g(z^{-1})}\right)\right\},$$

where $f(z) \in \Sigma_0$ and $g(z) \in \Sigma_{-1}$. It follows from (21) that $\theta_A$ is a homomorphism. Since $Ker\ \theta_A = \{[1, z]\}$, the map $\theta_A$ is one-to-one. In addition, for every $\{p_n(x)\} \in P_A$, from the definition of the generalized Sheffer-type polynomial sequence, there exists $[f,\, g] \in \bar{R}$ such that $\{p_n(x)\}$ is generated by $[f,\, g]$. Hence, $\theta_A$ is onto, which completes the proof of the theorem.

■

# 3 Subgroups of $\bar{\mathcal{R}}$ and $P_A$

Particular subgroups of $\mathcal{R}$ are important and have been considered in the literatures (see, for example, [15]):

- the set $\mathcal{A}$ of *Appell arrays*, that is the Riordan arrays $D = (d(z),\, h(z))$ for which $h(z) = z$; it is an invariant subgroup and is isomorphic to the group of f.p.s. of order 0, with the usual product as group operation;
- the set $\mathcal{L}$ of *Lagrange arrays*, that is the Riordan arrays $D = (d(z),\, h(z))$ for which $d(z) = 1$; it is also called the *associated subgroup*; it is isomorphic with the group of f.p.s. of order 1, with composition as group operation;
- the set $\mathcal{B}$ of *Bell* or *renewal arrays*, that is the Riordan arrays $D = (d(z),\, h(z))$ for which $h(z) = zd(z)$; it is the set originally considered by Rogers in [14];
- the set $\mathcal{H}$ of *hitting time arrays*, that is, the Riordan arrays $D = (d(z), h(z))$ for which $d(z) = \frac{zh'(z)}{h(z)}$; it is the subgroup with the usual Riordan product defined by Peart and Woan in [12].

We now extend the above subgroups to $\bar{\mathcal{R}}$ and $P_A$ as follows.
For $f \in \Sigma_0$ and $d \in \mathcal{F}_0$, since

$$[f(z),\, z] = \left(\frac{1}{f(z^{-1})}, z\right) \;\; and \;\; (d(z),\, z) = \left[\frac{1}{d(z^{-1})}, z\right],$$

by noting $1/f(z^{-1}) \in \mathcal{F}_0$ and $1/d(z^{-1}) \in \Sigma_0$, we know $\{[f(z),\, z] : f(z) \in \Sigma_0\}$ is one-to-one corresponds to the Appell subgroup of classical Riordan group. Hence, $\bar{\mathcal{A}} = \{[f(z),\, z] : f(z) \in \Sigma_0\}$ is a subgroup, called the Appell subgroup, of $\bar{\mathcal{R}}$, which is isomorphic to the group $\Sigma_0$ of Laurent series. The corresponding Appell subgroup of $P_A$ is the set of all polynomial sequences $\{p_n(x)\}$ defined by

$$\frac{1}{f(z^{-1})}A(xz) = \sum_{n\geq 0} p_n(x)z^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k x^k [z^n]\frac{z^k}{f(z^{-1})}\right) z^n,$$

or equivalently,

$$\frac{1}{f(z)}A(xz^{-1}) = \sum_{n\geq 0} p_n(x)z^{-n}.$$

Similarly, for $g \in \Sigma_1$, the set of all Riordan arrays $[1,\, g(z)]$, forms a subgroup denoted by $\bar{\mathcal{L}}$ and called the associated subgroup of $\bar{\mathcal{R}}$ because

$$[1,\, g(z)] = \left(1, \frac{1}{g(z^{-1})}\right) \; and \; (1, h(z)) = \left[1,\, \frac{1}{h(z^{-1})}\right].$$

The corresponding associated subgroup of $P_A$ is the set of all polynomial sequences $\{p_n(x)\}$ defined by

$$A\left(\frac{x}{g(z^{-1})}\right) = \sum_{n\geq 0} p_n(x) z^n = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k x^k [z^n] \frac{1}{(g(z^{-1}))^k}\right) z^n,$$

or equivalently,

$$A\left(\frac{x}{g(z)}\right) = \sum_{n\geq 0} p_n(x) z^{-n}.$$

Let $f \in \Sigma_0$ and $g \in \Sigma_{-1}$. Denote $d(z) = 1/f(z^{-1})$ and $h(z) = 1/g(z^{-1})$. Due to

$$\frac{1}{g(z^{-1})} = h(z) = z d(z) = \frac{1}{z^{-1} f(z^{-1})}$$

for all element $(d(z),\, h(z))$ in the Bell group of the classical Riordan group, it is reasonable to define the Bell subgroup of $\bar{\mathcal{R}}$ by $\bar{\mathcal{B}} = \{[f(z),\, zf(z)] : f(z) \in \Sigma_0\}$. And the corresponding Bell subgroup of $P_A$ is the set of all $\{p_n(x)\}$ defined by

$$\frac{1}{f(z^{-1})} A\left(\frac{xz}{f(z^{-1})}\right) = \sum_{n\geq 0} p_n(x) z^n = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k x^k [z^n] \frac{z^k}{(f(z^{-1}))^{k+1}}\right) z^n,$$

or equivalently,

$$\frac{1}{f(z)} A\left(\frac{x}{zf(z)}\right) = \sum_{n\geq 0} p_n(x) z^{-n}.$$

Similarly, we may define the hitting-time subgroup of $\bar{\mathcal{R}}$ as

$$\bar{\mathcal{H}} = \left\{ \left[\frac{g(z)}{zg'(z)},\, g(z)\right] : g(z) \in \Sigma_{-1} \right\}$$

and the hitting-time subgroup of $P_A$ by

$$\frac{z^{-1}g'(z^{-1})}{g(z^{-1})}A\left(\frac{x}{g(z^{-1})}\right)=\sum_{n\geq 0}p_n(x)z^n=\sum_{n\geq 0}\left(\sum_{k=0}^{n}a_kx^k[z^n]\frac{z^{-1}g'(z^{-1})}{(g(z^{-1}))^{k+1}}\right)z^n,$$

or equivalently,

$$\frac{zg'(z)}{g(z)}A\left(\frac{x}{g(z)}\right)=\sum_{n\geq 0}p_n(x)z^{-n}.$$

We survey the above results in the following theorem.

**Theorem 3.1** *Sets* $\bar{\mathcal{A}}=\{[f(z),\,z]:f\in\Sigma_0\}$, $\bar{\mathcal{L}}=\{[1,\,g(z)]:g\in\Sigma_{-1}\}$, $\bar{\mathcal{B}}=\{[f(z),\,zf(z)]:f\in\Sigma_0\}$, *and* $\bar{\mathcal{H}}=\{[g(z)/(zg'(z)),\,g(z)]:g\in\Sigma_{-1}\}$ *are subgroups of* $\bar{\mathcal{R}}$, *which are called, respectively, the Appell, associated, Bell, and hitting-time subgroups of* $\bar{\mathcal{R}}$.

*The sets of all polynomial sequences generated by all elements of* $\bar{\mathcal{A}}$, $\bar{\mathcal{L}}$, $\bar{\mathcal{B}}$, *and* $\bar{\mathcal{H}}$ *with respect to* $A(z)\in\mathcal{E}$ *are subgroups of* $P_A$ *and called the Appell, associated, Bell, and hitting-time subgroups of* $P_A$, *respectively.*

**Example 3.1** In Appell subgroup, let $f(z)=1-z^{-1}$ and $-1+z^{-1}$. Then $1/f(z^{-1})=1/(1-z)$ and $1/(z-1)$, respectively. For those $f(z)$, $[f(z),\,z]\in\bar{\mathcal{A}}$ and the corresponding Sheffer-type polynomial sequences $\{p_n(x)\}$ and $\{q_n(x)\}$ are defined, where $\{p_n(x)\}$ satisfies

$$\begin{aligned}\frac{1}{1-z}A\left(xz\right) &= \sum_{n\geq 0}p_n(x)=\sum_{n\geq 0}\left(\sum_{k=0}^{n}a_kx^k[z^n]\frac{z^k}{1-z}\right)z^n\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}a_kx^k\right)z^n.\end{aligned}$$

Thus, $p_n(x)=\sum_{k=0}^{n}a_kx^k$. Similarly, $q_n(x)=-\sum_{k=0}^{n}a_kx^k$.

**Example 3.2** Consider $f(z)=1$ and $g(z)=z$, $1-z$, and $z-1$, respectively. Thus, we have $1/g(z^{-1})=z$, $z/(z-1)$, and $z/(1-z)$, respectively. The corresponding generalized Sheffer-type polynomial sequences are, respectively $\{p_n(x)\}$, $\{q_n(x)\}$, and $\{r_n(x)\}$, where $p_n(x)$ are defined by

$$A\left(xz\right)=\sum_{n\geq 0}p_n(x)=\sum_{n\geq 0}a_nx^nz^n,$$

i.e., $p_n(x) = a_n x^n$, and, similarly, $q_n(x) = \sum_{k=0}^n (-1)^k a_k \binom{n-1}{n-k} x^k$ and $r_n(x) = \sum_{k=0}^n a_k \binom{n-1}{n-k} x^k$.

Another example is $[1, g(z)] \in \bar{\mathcal{L}}$, where $g(z) = 1/(\ln(1+z) - \ln z)$. Thus, $1/g(z^{-1}) = \ln(1+z)$, and the corresponding generalized Sheffer-type polynomial sequence $\{u_n(x)\}$ is defined by

$$A\left(x \ln(1+z)\right) = \sum_{n\geq 0} u_n(x) z^n = \sum_{n\geq 0} \left( \sum_{k=0}^n a_k x^k [z^n] \left(\ln(1+z)\right)^k \right) z^n.$$

In particular, if $A(z) = e^z$, then $a_n = 1/n!$ and $u_n(x) = (x)_n := x(x-1)(x-2)\cdots(x-n+1)$, the lower factorial polynomials.

**Example 3.3** In Bell subgroup, we consider $f(z) = 1 - z^{-1}$ and $z^{-1} - 1$. Thus, $[1 - z^{-1}, z - 1]$ and $[z^{-1} - 1, 1 - z] \in \bar{\mathcal{B}}$, and the corresponding generalized Sheffer-type polynomial sequence $\{p_n(x)\}$ for $f(z) = 1 - z^{-1}$ is

$$\begin{aligned} \frac{1}{1-z} A\left(\frac{xz}{1-z}\right) &= \sum_{n\geq 0} p_n(x) z^n = \sum_{n\geq 0} \left( \sum_{k=0}^n a_k x^k [z^n] \frac{z^k}{(1-z)^{k+1}} \right) z^n \\ &= \sum_{n\geq 0} \left( \sum_{k=0}^n a_k x^k \binom{n}{n-k} \right) z^n. \end{aligned}$$

Thus, $p_n(x) = \sum_{k=0}^n a_k \binom{n}{n-k} x^k$. Similarly, the generalized Sheffer-type polynomial sequence $\{q_n(x)\}$ for $f(z) = z^{-1} - 1$ is $q_n(x) = \sum_{k=0}^n (-1)^{k+1} a_k \binom{n}{n-k} x^k$. In particular, if $A(z) = 1/(1-z)$, then $p_n(x) = (1+x)^n$ and $q_n(x) = -(1-x)^n$.

**Example 3.4** In $[g(z)/(zg'(z)), g(z)]$, $g \in \Sigma_{-1}$, an element of the hitting-time subgroup $\bar{\mathcal{H}}$, let $g(z) = z/(z-1)$ and $z/(1-z)$. Then, $1/g(z^{-1}) = 1 - z$ and $z - 1$, and $g'(z^{-1}) = -z^2/(1-z)^2$ and $z^2/(z-1)^2$, respectively. The corresponding generalized Sheffer-type polynomial sequence $\{p_n(x)\}$ and $\{q_n(x)\}$ for $g(z) = z/(z-1)$ and $z/(1-z)$, respectively, can be defined by

$$\begin{aligned} -\frac{z}{1-z} A\left(x(1-z)\right) &= \sum_{n\geq 0} p_n(x) z^n = \sum_{n\geq 0} \left( \sum_{k=0}^n a_k x^k [z^n] (-z)(1-z)^{k-1} \right) \\ &= \sum_{n\geq 0} a_n (-x)^n \end{aligned}$$

and

$$\begin{aligned}\frac{z}{z-1}A\left(x(z-1)\right) &= \sum_{n\geq 0} q_n(x)z^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k x^k [z^n](z)(z-1)^{k-1}\right)\\ &= \sum_{n\geq 0} a_n x^n.\end{aligned}$$

Thus, $p_n(x) = \sum_{n\geq 0} a_n(-x)^n$ and $q_n(x) = \sum_{n\geq 0} a_n x^n$.

[3] showed that hitting-time subgroup of $\mathcal{R}$ is isomorphic to the Faber group formed by the set of all Faber polynomial sequences. Here, a Faber polynomial sequence $\{F_n(x)\}$ can be defined in two manners: in terms of Riordan arrays $(zh'(z)/h(z),\, h(z))$ associated with power series or Riordan arrays $[g(z)/(zg'(z)),\, g(z)]$ associated with Laurent series. From the instruction, we have

$$\frac{zh'(z)}{h(z)(1-xh(z))} = \sum_{n\geq 0} F_n(x)z^n$$

for some $h \in \mathcal{F}_1$, which implies that $\{F_n(x)\}$ is an ordinary Sheffer-type polynomial sequence generated by $(zh'(z)/h(z),\, h(z)) \in \mathcal{H}$ with respect to $A(z) = 1/(1-z)$. $\{F_n(x)\}$ can also be defined by

$$\frac{1}{f(z^{-1})}A\left(\frac{x}{g(z^{-1})}\right) \equiv \frac{g'(z)}{g(z)-x} = \sum_{n\geq 0}\frac{F_n(x)}{z^{n+1}},$$

where $A(z) = 1/(1-z)$, $f(z) = g(z)/(zg'(z))$, and $g(z) \in \Sigma_{-1}$. For example, let $g(z) = z - \rho$ ($\rho \in \mathbb{Z}$). Then the corresponding Faber polynomial sequence $F_n(x) = (\rho + x)^n$ because

$$\frac{g'(z)}{g(z)-x} = \frac{1}{z-\rho-x} = \frac{1}{z}\frac{1}{1-(\rho+x)/z} = \sum_{n\geq 0}\frac{(\rho+x)^n}{z^{n+1}}.$$

# 4 More Relationships between the Riordan arrays and generalized Sheffer-type polynomial sequences

For each $A \in \mathcal{E}$, Theorem 2.4 shows that the generalized Sheffer group $(P_A, \tilde{\#})$ associated with $A$ is isomorphic to the Riordan group $(\bar{R}, \#)$

associated with Laurent series. Hence, for $A, B \in \mathcal{E}$, groups $(P_A, \tilde{\#})$ and $(P_B, \tilde{\#})$ are isomorphic. We now establish a relationship between a generalized Sheffer-type polynomial sequence $\{\tilde{p}_n(x)\} \in P_A$ and the corresponding generalized Sheffer-type polynomial sequence $\{\tilde{q}_n(x)\} \in P_B$, where $\tilde{p}_n(x)$ and $\tilde{q}_n(x)$ are defined by (13) with the same $(d(z), h(z))$ and different power series $A$ and $B$, where $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$ are in $\mathcal{E}$. First, we have the following result.

**Theorem 4.1** *Let $A(z), B(z) \in \mathcal{E}$, and let $\{\tilde{p}_n(x)\} \in P_A$. Suppose $\{\tilde{p}_n^A(x)\}$ is generated by $(d(z), h(z))$ with respect to $A(z)$ and $\{\tilde{p}_n^B(x)\}$ is generated by the same $(d(z), h(z))$ with respect to $B(z)$ by using expansion (13). Then,*

$$\tilde{p}_n^A(x) = \sum_{k=0}^{n} \frac{a_k b_n}{a_n b_k} x^k \left([x^k]\tilde{p}_n^B(x)\right) \text{ and } \tilde{p}_n^B(x) = \sum_{k=0}^{n} \frac{b_k a_n}{b_n a_k} x^k \left([x^k]\tilde{p}_n^A(x)\right). \tag{22}$$

*In particular, if $A(z) = 1/(1-z)$, then*

$$\tilde{p}_n^A(x) = \sum_{k=0}^{n} \frac{b_n}{b_k} x^k \left([x^k]\tilde{p}_n^B(x)\right) \text{ and } \tilde{p}_n^B(x) = \sum_{k=0}^{n} \frac{b_k}{b_n} x^k \left([x^k]\tilde{p}_n^A(x)\right). \tag{23}$$

*Proof.* To prove (22), it is sufficient to note

$$\tilde{p}_n^A(x) = \sum_{k=0}^{n} \frac{a_k}{a_n} x^k [z^n] d(z)(h(z))^k \text{ and } \tilde{p}_n^B(x) = \sum_{k=0}^{n} \frac{b_k}{b_n} x^k [z^n] d(z)(h(z))^k.$$

(23) immediately follows when $a_n = 1$ for all $n \geq 0$.

■

**Example 4.1** It is well-known that Bernoulli polynomial sequence is generated by

$$\frac{z}{e^z - 1} e^{xz} = \sum_{n\geq 0} \frac{1}{n!} B_n(x) z^n.$$

Noting the well-know expansion

$$\frac{z}{e^z - 1} = \sum_{\ell \geq 0} \frac{B_\ell}{\ell!} z^\ell,$$

where $B_\ell$ is the $\ell$th Bernoulli number, we obtain that the ordinary polynomial sequence $\{p_n(x)\}$ defined by the same Riordan array $(z/(e^z-1),\, z)$ but with respect to $A(z) = 1/(1-z)$ can be presented as

$$\begin{aligned}\frac{z}{e^z-1}\frac{1}{1-xz} &= \sum_{n\geq 0} p_n(x)z^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} x^k[z^n]\frac{z}{e^z-1}z^k\right)z^n \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} x^k\frac{B_{n-k}}{(n-k)!}\right)z^n.\end{aligned}$$

Thus, from (23) we obtain the explicit formula of Bernoulli polynomials

$$B_n(x) = \sum_{k=0}^{n}\frac{n!}{k!}\frac{B_{n-k}}{(n-k)!}x^k = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

It is known that $(p-1)$st order Laguerre polynomial sequence $\{L_n^{(p-1)}(x)\}$ is generated by involution $(1/(1-z)^p,\, z/(z-1))$ with respect to $B(z) = e^z$,

$$\frac{1}{(1-z)^p}e^{xz/(z-1)} = \sum_{n\geq 0} L_n^{(p-1)}(x)z^n.$$

Consider the ordinary polynomial sequence $\{q_n(x)\}$ generated by

$$\begin{aligned}\frac{1}{(1-z)^p}\frac{1}{1-xz/(z-1)} &= \sum_{n\geq 0} q_n(x)z^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n}(-x)^k[z^n]\frac{z^k}{(1-z)^{p+k}}\right)z^n \\ &= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n+p-1}{n-k}(-x)^k.\end{aligned}$$

Thus, formula (22) yields

$$L_n^{(p-1)}(x) = \sum_{k=0}^{n}\frac{1}{k!}\binom{n+p-1}{n-k}(-x)^k,$$

where the case $p = 1$ was shown in [3].

Angelescu polynomial sequence $\{A_n(x)\}$ is a Sheffer-type polynomial sequence defined by

$$\frac{1}{1+z}e^{xz/(z-1)} = \sum_{n\geq 0} A_n(x)z^n.$$

Consider the ordinary Sheffer-type polynomial sequence $\{r_n(x)\}$ generalized by the same Riordan array $(1/(1+z),\, z/(z-1))$ with respect to $A(z) = 1/(1-z)$:

$$\begin{aligned}
&\frac{1}{1+z}\frac{1}{1-xz/(z-1)} = \sum_{n\geq 0} r_n(x) z^n \\
= &\sum_{n\geq 0}\left(\sum_{k=0}^{n} x^k [z^n] \frac{1}{1+z}\left(\frac{z}{z-1}\right)^k\right) \\
= &\sum_{n\geq 0}\left(\sum_{k=0}^{n} (-x)^k [z^n]\left(\sum_{j\geq 0}(-z)^j \sum_{\ell\geq 0}\binom{\ell+k-1}{\ell} z^{k+\ell}\right)\right) \\
= &\sum_{n\geq 0}\left(\sum_{k=0}^{n} (-x)^k [z^n]\left(\sum_{j\geq 0}\sum_{\ell=0}^{j}(-1)^{j-\ell}\binom{\ell+k-1}{\ell} z^{j+k}\right)\right) \\
= &\sum_{n\geq 0}\sum_{k=0}^{n}\left(\sum_{\ell=0}^{n-k}(-1)^{n-\ell}\binom{\ell+k-1}{\ell}\right) x^k.
\end{aligned}$$

Thus, (22) gives an explicit formula of Angelescu polynomials

$$A_n(x) = (-1)^n \sum_{k=0}^{n} \frac{1}{k!}\left(\sum_{\ell=0}^{n-k}(-1)^{\ell}\binom{\ell+k-1}{\ell}\right) x^k.$$

We now define the Riordan pairs and generalized Stirling number pairs.

**Definition 4.2** *Let $d(z) \in \mathcal{F}_0$, $h(z) \in \mathcal{F}_1$, $f(z) \in \Sigma_0$, and $g(z) \in \Sigma_{-1}$. Then the Riordan pairs $\{d_{n,k}, \tilde{d}_{n,k}\}$ generated by $(d(z),\, h(z))$ and $[f(z),\, g(z)]$ are defined by, respectively,*

$$d(z)(h(z))^k = \sum_{n\geq k} d_{n,k} z^n, \quad d(\bar{h}(z))^{-1}(\bar{h}(z))^k = \sum_{n\geq k} \tilde{d}_{n,k} z^n, \tag{24}$$

*and*

$$\frac{1}{f(z^{-1})}\left(\frac{1}{g(z^{-1})}\right)^k = \sum_{n\geq k} d_{n,k} z^n,\ f(\bar{g}(z^{-1}))\left(\frac{1}{\bar{g}(z^{-1})}\right)^k = \sum_{n\geq k} \tilde{d}_{n,k} z^n, \tag{25}$$

*where $\bar{h}(z)$ is the compositional inverse of $h(z)$ in terms of $z$, i.e., $\bar{h}(h(z)) = h(\bar{h}(z)) = z$, and $\bar{g}(z^{-1})$ is the compositional inverse of $g(z^{-1})$ in terms of $z^{-1}$, i.e., $\bar{g}(g(z^{-1})) = z^{-1}$ and $g(\bar{g}(z^{-1})) = z^{-1}$.*

Following the definition of generalized Stirling numbers shown in the instruction, we have their pairs defined below.

**Definition 4.3** *Let $d(z) \in \mathcal{F}_0$, $h(z) \in \mathcal{F}_1$, $f(z) \in \Sigma_0$, and $g(z) \in \Sigma_{-1}$. Then the generalized Stirling number pairs $\{\sigma_{n,k}, \tilde{\sigma}_{n,k}\}$ generated by $(d(z), h(z))$ and $[f(z), g(z)]$ with respect to $A(z) \in \mathcal{E}$ are defined by, respectively,*

$$d(z)(h(z))^k = \sum_{n\geq k} \frac{a_n}{a_k}\sigma_{n,k}z^n, \quad d(\bar{h}(z))^{-1}(\bar{h}(z))^k = \sum_{n\geq k} \frac{a_n}{a_k}\tilde{\sigma}_{n,k}z^n, \quad (26)$$

*and*

$$\frac{1}{f(z^{-1})}\left(\frac{1}{g(z^{-1})}\right)^k = \sum_{n\geq k} \frac{a_n}{a_k}\sigma_{n,k}z^n, \ f(\bar{g}(z^{-1}))\left(\frac{1}{\bar{g}(z^{-1})}\right)^k = \sum_{n\geq k} \frac{a_n}{a_k}\tilde{\sigma}_{n,k}z^n, \quad (27)$$

*where $\bar{h}(z)$ is the compositional inverse of $h(z)$ in terms of $z$, and $\bar{g}(z^{-1})$ is the compositional inverse of $g(z^{-1})$ in terms of $z^{-1}$.*

Similar to [6], we may use the orthogonality of the Riordan pairs and generalized Stirling number pairs to give several inverse relationships of power series.

**Theorem 4.4** *Let $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be two sequences. Then there exist two inverse relationships between them which are generalized by using the Riordan pairs and generalized Stirling number pairs shown in Definitions 4.2 and 4.3. Namely, the existence of one formula below implies the existence of another one.*

$$a_n f_n = \sum_{k=0}^{n} a_k d_{n,k} g_k \iff a_n g_n = \sum_{k=0}^{n} a_k \tilde{d}_{n,k} f_k \quad (28)$$

$$f_n = \sum_{k=0}^{n} \sigma_{n,k} g_k \iff g_n = \sum_{k=0}^{n} \tilde{\sigma}_{n,k} f_k. \quad (29)$$

*Proof.* It is sufficient to show that Definitions 4.2 and 4.3 implies

$$\sum_{n\geq k\geq \ell} d_{n,k}\tilde{d}_{k,\ell} = \sum_{n\geq k\geq \ell} \tilde{d}_{n,k} d_{k,\ell} = \delta_{n,\ell}$$

and

$$\sum_{n\geq k\geq \ell} \sigma_{n,k}\tilde{\sigma}_{k,\ell} = \sum_{n\geq k\geq \ell} \tilde{\sigma}_{n,k}\sigma_{k,\ell} = \delta_{n,\ell},$$

where $\delta_{n,\ell}$ is the Kronecker symbol.

■

From (14) and the orthogonality shown above, we can see the equivalence between the pair of generalized Stirling numbers and the corresponding pairs of Riordan arrays.

From (13) in the instruction, we know that the generalized Sheffer-type polynomial sequences related to the generalized Stirling numbers $\sigma(n,k)$ and $\tilde{\sigma}(n,k)$ are given respectively by the following expressions with respect to $A(Z) \in \mathcal{E}$.

$$\frac{1}{a_n} p_n(x) = \sum_{k=0}^{n} \sigma(n,k) x^k \tag{30}$$

and

$$\frac{1}{a_n} \bar{p}_n(x) = \sum_{k=0}^{n} \bar{\sigma}(n,k) x^k, \tag{31}$$

where $p_n(x)$ and $\bar{p}_n(x)$ are generalized Sheffer-type polynomials associated with $(d(z),\, h(z))$ and $(d(\bar{h}(z))^{-1}, \bar{h}(z))$, respectively, or $[f(z),\, g(z)]$ and $[f(\bar{g}(z))^{-1},\, \bar{g}(z)]$, respectively. Here $d \in \mathcal{F}_0$, $h \in \mathcal{F}_1$, $f \in \Sigma_0$, and $g \in \Sigma_{-1}$, $\bar{h}$ is the compositional inverse of $h$ in terms of $z$, and $\bar{g}$ is the compositional inverse of $g$ in terms of $z^{-1}$. We call $\{p_n(x), \bar{p}_n(x)\}$ the pair of generalized Sheffer-type polynomial sequences generated by $(d(z),\, h(z))$ or $[f(z),\, g(z)]$ with respect to $A(z)$.

Applying the reciprocal relations (28) to (30)-(31) we get

**Corollary 4.5** *There hold the relations*

$$\sum_{k=0}^{n} \bar{\sigma}(n,k) \frac{1}{a_k} p_k(x) = x^n \tag{32}$$

*and*

$$\sum_{k=0}^{n} \sigma(n,k) \frac{1}{a_k} \bar{p}_k(x) = x^n. \tag{33}$$

*or equivalently,*

$$\sum_{k=0}^{n} \bar{d}_{n,k} p_k(x) = a_n x^n \quad \textit{and} \quad \sum_{k=0}^{n} d_{n,k} \bar{p}_k(x) = a_n x^n. \tag{34}$$

**Example 4.2** Suppose $f \in \Sigma_0$ and $g \in \Sigma_{-1}$ and denote $d(z) = 1/f(z^{-1})$ and $h(z) = 1/g(z^{-1})$. Thus $d(z) \in \mathcal{F}_0$ and $h(z) \in \mathcal{F}_1$. Example 1.1 gives the ordinary Sheffer-type polynomial sequence $\{p_n(x)\}$ generated by (15) and (16). Thus,

$$p_n(x) = \sum_{k=0}^{n} d_{n,k} x^k = \sum_{k=0}^{n} [z^n] d(z)(h(z))^k x^k,$$

and the corresponding Sheffer-type polynomial sequence pair $\{p_n(x), \bar{p}_n(x)\}$ is given by the above expression and

$$\bar{p}_n(x) = \sum_{k=0}^{n} \tilde{d}_{n,k} x^k = \sum_{k=0}^{n} [z^n] d(\bar{h}(z))^{-1} (\bar{h}(z))^k x^k.$$

If $d(z) = \frac{zh'(z)}{h(z)}$, or equivalently, $f(z) = \frac{g(z)}{zg'(z)}$, then the corresponding $\{F_n(x), \bar{F}_n(x)\}$ defined by

$$\begin{aligned} F_n(x) &= \sum_{k=0}^{n} d_{n,k} x^k = \sum_{k=0}^{n} [z^{n-1}] h'(z)(h(z))^{k-1} x^k \\ \bar{F}_n(x) &= \sum_{k=0}^{n} \tilde{d}_{n,k} x^k = \sum_{k=0}^{n} [z^{n-1}] \bar{h}'(z)(\bar{h}(z))^{k-1} x^k, \end{aligned}$$

where $\bar{h}$ is the compositional inverse of $h$ in terms of $z$, and $\{F_n(x)\}$ is the Faber polynomial sequence. From Corollary 4.5, we obtain the following identities

$$\begin{aligned} \sum_{k=0}^{n} \left( [z^{n-1}] \bar{h}'(z)(\bar{h}(z))^{k-1} \right) F_k(x) &= x^n \\ \sum_{k=0}^{n} \left( [z^{n-1}] h'(z)(h(z))^{k-1} \right) \bar{F}_k(x) &= x^n. \end{aligned}$$

For instance, if $h(z) = z/(1-z)$, then $\{F_k(x)\}$ is the zero order Laguerre polynomial sequence $\{L_n(x)\}$. Thus the above identities can be specified to

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} L_n(x) = x^n.$$

Here, we use the fact of $\bar{h}(z) = h(z)$ and $\bar{L}_n(x) = L_n(x)$.

## Acknowledgments

We wish to thank the editor and referees for their helpful and valuable comments and suggestions.

## References

[1] A. Z. Broder, The $r$-Stirling numbers, *Discrete Math.*, 49(1984), 241-259, North Holland

[2] G.-S. Cheon, H. Kim, L. W. Shapiro, Riordan group involutions. *Linear Algebra Appl.*, 428 (2008), 941–952.

[3] G.-S. Cheon, H. Kim, L. W. Shapiro, An algebraic structure for Faber polynomials, *Linear Algebra Appl.*, 433 (2010), 1170 -1179.

[4] C. Corsani, D. Merlini, and R. Sprugnoli. Left inversion of combinatorial sums. *Discrete Math.*, 180 (1998), 107–122.

[5] T. X. He, A Symbolic Operator Approach to Power Series Transformation-Expansion Formulas. *J. Integer Sequences*, 11(2008), No. 2, Article 08.2.7, 1-19.

[6] T. X. He, L. C. Hsu, P. J.-S. Shiue. The Sheffer Group and the Riordan Group. *Discrete Applied Math.*, 155 (2007), 1895–1909.

[7] T. X. He and R. Sprugnoli. Sequence Characterization of Riordan Arrays, *Discrete Math.*, 309 (2009), 3962-3974.

[8] F. Howard, Degenerate weighted Stirling numbers, *Discrete Math.*, 57(1985), 45-58.

[9] L. C. Hsu, Theory and application of general Stirling number pairs, *J. Math. Res. & Exposition*, 9(1989), no. 2, 211-220.

[10] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. On some alternative characterizations of Riordan arrays. *Canadian J. Mathematics*, 49 (1997), 301–320.

[11] A. Nkwanta. A Riordan matrix approach to unifying a selected class of combinatorial arrays. *Congressus Numerantium*, 160 (2003), 33–45.

[12] P. Peart, and W.-J. Woan. A divisibility property for a subgroup of Riordan matrices. *Discrete Applied Math.*, 98 (2000), 255–263.

[13] J. Riordan, *An Introduction to Combinatorial Analysis,* New York, Wiley, 1958.

[14] D. G. Rogers. Pascal triangles, Catalan numbers and renewal arrays. *Discrete Math.*, 22 (1978), 301–310.

[15] L. W. Shapiro. A survey of the Riordan Group. Talk at a meeting of the American Mathematical Society, Richmond, Virginia, 1994.

[16] L. W. Shapiro. Some open questions about random walks, involutions, limiting distributions and generating functions, *Advances in Applied Math.*, 27 (2001), 585–596.

[17] L. W. Shapiro. Bijections and the Riordan group. *Theoretical Computer Science*, 307 (2003), 403–413.

[18] L. W. Shapiro, S. Getu, W.-J. Woan, and L. Woodson. The Riordan group. *Discrete Applied Math.*, 34 (1991), 229–239.

[19] M. Schiffer, Faber polynomials in the theory of univalent functions, *Bull. Amer. Math. Soc.* 54 (1948), 503-517.

[20] R. Sprugnoli. Riordan arrays and combinatorial sums. *Discrete Math.*, 132 (1994), 267–290.

[21] R. Sprugnoli. Riordan arrays and the Abel-Gould identity. *Discrete Math.*, 142 (1995), 213–233.

[22] W. Wang and T. Wang, Generalized Riordan arrays, *Discrete Math.* 308 (2008), 6466–6500.