Double Riordan Arrays and Their Compressions

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Fourth International Symposium
on Riordan Arrays and Related Topics
Universidad Complutense de Madrid in Madrid
17-20 July 2017

Abstract

Inspired by the Fibonacci tree shown in the recent book, Catalan Numbers, by Richard Stanley, we present a combinatorial way to construct double Riordan arrays by using ECO technique. The sequence characterizations of double Riordan arrays and subgroups of the double Riordan group are found. As an extension of Pascal-Fibonacci array, the compressions of double Riordan arrays are defined. Some connections of Riodan arrays and double Riordan arrays as well as their compressions are given. The results are also extended to high order Riordan arrays. The properties of double Riordan arrays and their compressions such as P matrices, stabilizers, Sheffer polynomial sequence pairs, etc. are discussed.

Double Riordan Arrays and Their Compressions

- What are Double Riordan Arrays?
- Rooted trees, Enumeration of combinatorial objects (ECO), and Riordan arrays
- Succession rules and compressions of double Riordan arrays
- Sequence characterization of double Riordan arrays
- Double Riordan arrays and their compressions
- High order Riordan arrays and their compressions
- Stabilizer subgroups of the double Riordan group
- P matrices of the double Riordan arrays and their compressions
- Sheffer sequence pairs and double Riordan arrays

Riordan arrays and Riodan group

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group*.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = K[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ $(f_k \in \mathbb{R})$, is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k.

Riordan arrays and Riodan group

From the *fundamental theorem of Riordan arrays*, it is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t))(d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \quad (2)$$

The Riordan array $I = (1,\, t)$ acts as an identity for this product.

- the set $\mathcal{A}$ of *Appell arrays*: $R = (d(t),\, t)$ in $\mathcal{R}$;
- the set $\mathcal{L}$ of *Lagrange arrays*: $R = (1,\, h(t))$ in $\mathcal{R}$;
- the set $\mathcal{B}$ of *Bell* or *renewal arrays*: $R = (d(t),\, td(t))$ in $\mathcal{R}$;
- the set $\mathcal{H}$ of *hitting-time arrays*: $R = (th'(t)/h(t),\, h(t))$ in $\mathcal{R}$;
- *derivative subgroup*: the set $\mathcal{D}$ of $R = (h'(t),\, h(t))$ in $\mathcal{R}$;
- *Luźon-Merlini-Morón-Sprugnoli (LMMS) subgroup*: the set $\mathcal{E}$ of $R = ((h(t)/t)^r h'(t)^s, h(t))$ for real or complex r and s in $\mathcal{R}$, which includes $\mathcal{D} = \mathcal{E}[0, 1]$ as its special case.

Riordan arrays and Riodan group

An infinite lower triangular array $[d_{n,k}]_{n,k\in\mathbb{N}} = (d(t), h(t))$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n}. \tag{3}$$

This is equivalent to

$$h(t) = tA(h(t)) \quad \text{or} \quad t = \bar{h}(t)A(t). \tag{4}$$

Here, $A(t)$ is the generating function of the A-sequence. There exists a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{5}$$

or equivalently,

$$d(t) = \frac{d_{0,0}}{1 - tZ(h(t))}. \tag{6}$$

Riordan arrays and Riodan group

Denote the *upper shift matrix* by U, i.e.,

$$U = (\delta_{i+1,j})_{i,j\geq 0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \textit{and}$$

Riordan arrays and Riodan group

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left(Z(t), A(t), tA(t), t^2A(t), \ldots\right), \tag{7}$$

where the rightmost expression is the representation of P by using its column generating functions. Here, P is called the *production matrix* or *P- matrix characterization* or simply *P matrix*. The P-matrix of Riordan array R satisfies

$$P = R^{-1}UR. \tag{8}$$

ECO method related to the succession rule

The ECO method is a constructive method to produce all the objects of a given class, according to the growth of a certain parameter (in terms of the size) of the objects. A complete description of the ECO method and its applications for the enumeration of several classes of combinatorial objects is given in Barcucci, Del Lungo, Pergola, and Pinzani in 1999. The roots of the ECO method can be traced back to the paper by Chung, Graham, Hoggatt, and Kleiman in 1978 using Baxter permutation.

ECO method related to the succession rule

If an ECO construction is sufficiently regular, then it is often possible to describe it using a succession rule, whose definition is first introduced by Julian West in 1996. Some algebraic properties of succession rules have been determined in Ferrari, Pergola, Pinzani, and Rinaldi, 2002. Succession rules play an important role in the general framework of the ECO method: in practice, they are the tool to translate the recursive construction given by the ECO operator, onto a system of functional equation satisfied by the generating function of the studied class. Intimately related to the concept of succession rule is the notion of a generating tree, which is the most common way of representing a succession rule.

ECO method related to the succession rule

A *succession rule* is a formal system consisting of an axiom (a), $a \in \mathbb{N}_+$, and a set of productions

$$\{(k_j) \to (e_1(k_j))(e_2(k_j)) \cdots (e_j(k_j)) : j \in \mathbb{N}\},$$

where $e_i : \mathbb{N}_+ \to \mathbb{N}_+$, for deriving the successors $(e_1(k))(e_2(k)) \cdots (e_k(k))$ of any given label (k), $k \in \mathbb{N}$. Thus, we may write a succession rule, denoted by Ω, as

$$\Omega : \begin{cases} (a) \\ (k) \to (e_1(k))(e_2(k)) \cdots (e_k(k)). \end{cases} \tag{9}$$

$(a), (k)$, and $(e_i(k))$ $(a, k, e_i(k) \in \mathbb{N}_+)$ are called the labels of Ω. The succession rule can be presented as a *generating rooted tree* called a *generating tree*, whose vertices are the labels of Ω, where (a) is the label of the root, and each node labeled (k) has k' sons labeled by $(e_1(k)), \ldots, e_{k'}(k))$, respectively. Denote by f_n the number of nodes at level n in the generating tree determine by Ω.

ECO method related to the succession rule

We call $\{f_n\}_{n\geq 0}$ the sequence $\{f_n\}_{n\geq 0}$ associated with Ω. In Ferrari, Pergola, Pinzani, and Rinaldi, 2005, they represent the succession rule Ω by an infinite matrix $P = (p_{i,j})_{i,j\geq 0}$, called the *production matrix* of the succession rule Ω, where $p_{i,j}$ is the number of labels ℓ_j produced by label ℓ_i, and $\{(\ell_k)\}_{k\geq 0}$ is the label set of Ω with the axiom label ℓ_0. Simply speaking, if Ω containing root (a) satisfies succession rule
$(a) \to (a)\dots(a)(a+1)\dots(a+1)\dots(a+m)\dots(a+m)$, which we shorten to $(a)^{n_1}(a+1)^{n_2}\dots(a+m)^{n_m}$, then the first row of P is $(n_1, n_2, \cdots, n_m, 0, \cdots)$, where n_j $(j = 1, 2, \dots, m)$ may be zero, say $n_k = 0$, if $(a+k)$ is not a label. Similarly, we may write other rows of P row by row.

ECO method related to the succession rule

In Deutsch, Ferrari, and Rinaldi, 2009, the following result is proved, which can be viewed as a particular case of (8) when A- and Z- sequences are non-negative sequences. The case of not non-negative A- and/or Z-sequences is discussed in H, 2015.

Proposition

Let P be an infinite production matrix and let A_P be the ECO matrix induced by P defined before. Then A_P is a Riordan matrix if and only if P is of the form

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (10)$$

ECO method related to the succession rule

Let us consider the Fibonacci tree F_1 shown in Horibe, 1983:

$$\Omega : \begin{cases} (a) \\ a(k) \to a(k)a(k+1). \end{cases}$$

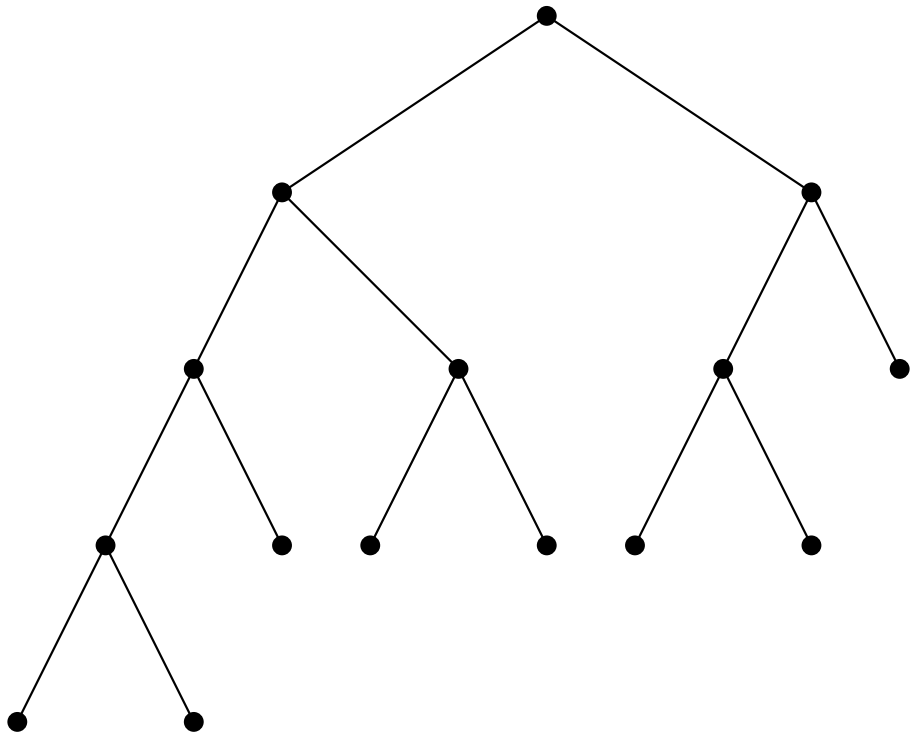


Figure: Figure 1. Fibonacci tree F_1

ECO method related to the succession rule

The P matrix of the succession rule is

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{11}$$

and the corresponding Riordan array (1/(1-t), t/(1-t)), the pascal matrix, is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{12}$$

The sum of the numbers along the parallel lines in the direction of $(1, 1)$ are the Fibonacci numbers, $\{1, 1, 2, 3, 5, 8, \ldots\}$.

ECO method related to the succession rule

Succession rule can generates more wide class of lower triangle matrices that do not belong to Riordan array class. for instance, in Stanley, 2015, the Fibonacci tree F_2 is defined to be the rooted tree with root v, such that the root has degree one, the child of every vertex of degree one has degree two, and the two children of every vertex of degree two have degrees one and two. Figure 2 shows the first six levels of F_2.

ECO method related to the succession rule

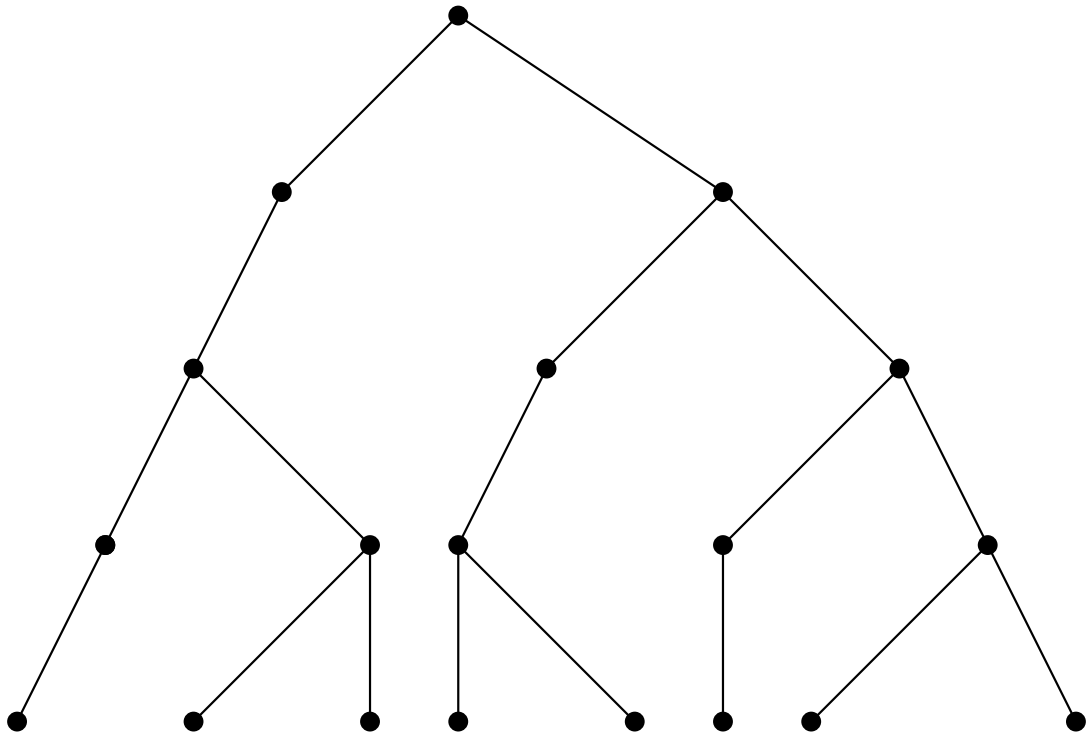


Figure: Figure 2. Fibonacci tree F_2

ECO method related to the succession rule

The succession rule Ω for the Fibonacci tree is

$$\Omega : \begin{cases} (a_{2l}) \to (a_{2l})(a_{2l+1}) \\ (a_{2l+1}) \to (a_{2l+2}), \end{cases} \tag{13}$$

for $\ell = 0, 1, 2, \ldots$. This succession rule generates the following lower triangle matrix $\bar{R}$:

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 3 & 2 & 1 & 0 & \ldots \\ 1 & 1 & 4 & 3 & 3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{14}$$

Matrix $\hat{R}$ is called *Fibonacci-Stanley array*. The row sums of $\hat{R}$ are Fibonacci numbers. Obviously, $\hat{R}$ is not a Riordan array. We will see it is the compression of a double Riordan array.

ECO method related to the succession rule

In the k^{th} column when $k = 2l, l > 0$ we have the recurrence

$$d_{n,k} = d_{n-1,k-1} + d_{n-1,k},$$

which implies that $g(f_1 f_2)^k = zg(f_1 f_2)^{k-1} f_1 + zg(f_1 f_2)^k$ Hence,

$$f_2 = z/(1-z).$$

When $k = 2l + 1$ we have the recurrence

$$d_{n,k} = d_{n-2,k-2} + d_{n-2,k-1},$$

which implies that $g(f_1 f_2)^k f_1 = z^2 g(f_1 f_2)^{k-1} f_1 + z^2 g(f_1 f_2)^k$ Hence,

$$f_1 f_2 = z^2 + z^2 f_2.$$

Thus, we have $f_1 = z$. By observation, the generating function of the first column of R can be written as $g(z) = 1/(1-z)$. Hence, $\hat{R} = (g; f_1, f_2) = (\frac{1}{1-z}, z, z/(1-z))$ forms the compression of the double Riordan array presenting the Fibonacci tree F.

Double Riordan arrays

In construction of a Riordan array, one multiplier function h is used to multiply one column to obtain the next column. Suppose alternating rules are applied to generate an infinite matrix similar to a Riordan array. To consider this case, one may use two multiplier functions, denoted by h_1 and h_2, respectively. Davenport, Shapiro, and Woodson (2012) require that d to be an even function and h_1 and h_2 to be odd functions so that they can develop an analog of the fundamental theorem and thus obtain a group structure. Let $d(t) = \sum_{k=0}^{\infty} d_{2k} t^{2k}$, $h_1(t) = \sum_{k=0}^{\infty} h_{1,2k+1} t^{2k+1}$, and $h_2(t) = \sum_{k=0}^{\infty} h_{2,2k+1} t^{2k+1}$. Then the double Riordan matrix in terms of $d(t)$, $h_1(t)$, and $h_2(t)$, denoted by $(d; h_1, h_2)$, is defined by the generating function of its columns as

$$(d,\ dh_1,\ dh_1h_2,\ dh_1^2h_2,\ dh_1^2h_2^2,\ldots).$$

Double Riordan arrays

There two cases of the *first fundamental theorem of double Riordan arrays*:

$$(d; h_1, h_2)A(t) = B(t), \tag{15}$$

where for $A(t) = \sum_{k\geq 0} a_{2k}t^{2k}$ and $A(t) = \sum_{k\geq 0} a_{2k+1}t^{2k+1}$, we have $B(t) = dA(\sqrt{h_1h_2})$ and $B(t) = d\sqrt{h_1/h_2}A(\sqrt{h_1h_2})$, respectively. Based on the fundamental theorem of double Riordan arrays, we may define a multiplication of two double Riordan arrays as

$$\begin{aligned} & (d; h_1, h_2)(g; f_1, f_2) \\ = \quad & (dg(\sqrt{h_1h_2}); \sqrt{h_1/h_2}f_1(\sqrt{h_1h_2}), \\ & \sqrt{h_2/h_1}f_2(\sqrt{h_1h_2})), \end{aligned} \tag{16}$$

where $g(t) = \sum_{k=0}^{\infty} g_{2k}t^{2k}$, $f_1(t) = \sum_{k=0}^{\infty} f_{1,2k+1}t^{2k+1}$, and $f_2(t) = \sum_{k=0}^{\infty} f_{2,2k+1}\ t^{2k+1}$.

Double Riordan arrays

The collection of all double Riordan arrays associated with the multiplication defined above form a group called double Riordan group and denoted by $\mathcal{DR}$, where the identity is $(1, t, t)$.
$\mathcal{S} := \{(d; h_1, h_2) \in \mathcal{DR} : d = 1\}$,
$\mathcal{B}_1 := \{(d; h_1, h_2) \in \mathcal{DR} : h_1 = td\}$,
$\mathcal{B}_2 := \{(d; h_1, h_2) \in \mathcal{DR} : h_2 = td\}$, and $\mathcal{A} := \{(d; t, t) \in \mathcal{DR}\}$ are associate, type-1 Bell, type-2 Bell, and Appel subgroups of $\mathcal{DR}$, respectively. The row sum generating function of a double Riordan array $(d; h_1, h_2)$ is

$$\begin{aligned} (d; h_1, h_2)\frac{1}{1-t} &= d + dh_1 + dh_1h_2 + dh_1^2h_2 + dh_1^2h_2^2 + \cdots \\ &= d(1 + (h_1h_2) + (h_1h_2)^2 + \cdots) + dh_1(1 + (h_1h_2) + (h_1h_2)^2 + \cdots) \\ &= \frac{d}{1-h_1h_2} + \frac{dh_1}{1-h_1h_2} = \frac{d(1+h_1)}{1-h_1h_2}. \end{aligned} \tag{17}$$

Double Riordan arrays

$\mathcal{A}$ is a normal subgroup of $\mathcal{DR}$ and $\mathcal{DR}$ is the semidirect product of $\mathcal{A}$ and $\mathcal{S}$. If $(d; h_1, h_2)$ is a double Riordan array, where

$$d(t) = \sum_{k=0}^{\infty} d_{2k} t^{2k},$$

$$h_1(t) = \sum_{k=0}^{\infty} h_{1,2k+1} t^{2k+1},$$

$$h_2(t) = \sum_{k=0}^{\infty} h_{2,2k+1} t^{2k+1}.$$

Then $(\hat{d}; \hat{h}_1, \hat{h}_2)$, where $\hat{d}(t) = \sum_{k=0}^{\infty} d_{2k} t^k$, $\hat{h}_1(t) = \sum_{k=0}^{\infty} h_{1,2k+1} t^{k+1}$, and $\hat{h}_2(t) = \sum_{k=0}^{\infty} h_{2,2k+1} t^{k+1}$, is the compression of the double Riordan array $(d; h_1, h_2)$.

Double Riordan arrays

Hence, the Pascal-Fibonacci array $\hat{R} = \left(\frac{1}{1-t}, t, \frac{t}{1-t}\right)$ shown above is the compression of the Double Riordan array $\left(\frac{1}{1-t^2}, t, \frac{t}{1-t^2}\right)$.

Sequence characterization of double Riordan array

Theorem

$(d; h_1, h_2) = (d_{n,k})_{n,k\geq 0}$ is in $\mathcal{DR}$ with $d_{n,n} \neq 0$ if and only if there exist A_1-sequence, $\{a_{1,0}, a_{1,1}, a_{1,2}, \ldots\}$, A_2- sequence, $\{a_{2,0} \neq 0, a_{2,1}, a_{2,2}, \ldots\}$, and Z-sequence, $\{z_0, z_1, z_2, \ldots\}$ such that

$$\begin{aligned} d_{n,2k-1} &= a_{1,0}d_{n-1,2k-2} + a_{1,1}d_{n-1,2k} + a_{1,2}d_{n-1,2k+2} + \cdots, \quad k \geq 1 \\ d_{n,2k} &= a_{2,0}d_{n-2,2k-2} + a_{2,1}d_{n-2,2k} + a_{2,2}d_{n-2,2k+2} + \cdots, \quad k \geq 1 \\ d_{n,0} &= z_0 d_{n-2,0} + z_1 d_{n-2,2} + z_2 d_{n-2,4} + \cdots \end{aligned} \tag{18}$$

for $n \geq 0$.

Sequence characterization of double Riordan array

Furthermore, denoting the generating functions of A_1-sequence, A_2-sequence, and Z-sequence by $A_1(t) = \sum_{k\geq 0} a_{1,k} t^{2k}$, $A_2(t) = \sum_{k\geq 0} a_{2,k} t^{2k}$, and $Z(t) = \sum_{k\geq 0} z_k t^{2k}$, respectively, we have an equivalent form of (18)

$$h_1 = tA_1(\sqrt{h_1 h_2}), \tag{19}$$

$$h_1 h_2 = t^2 A_2(\sqrt{h_1 h_2}), \tag{20}$$

$$d(t) = \frac{d(0)}{1 - t^2 Z(\sqrt{h_1 h_2})}. \tag{21}$$

Sequence characterization of double Riordan array

Example 2.1 $\hat{R}$ shown in (39) is the compression of the double Riordan array $(1/(1-t^2), t, t/(1-t^2))$, which presents the Fibonacci tree shown in Figure 2. Thus $h_1 = t$, $h_2 = t/(1-t^2)$, and $d = 1/(1-t^2)$. and $\sqrt{h_1 h_2} = t/\sqrt{1-t^2}$. The compositional inverse of $\sqrt{h_1 h_2}$ is $t/\sqrt{1+t^2}$. Substituting $h_1 = t$ and $d = 1/(1-t^2)$ into Equations (19) and (21), we immediately have $A_1 = 1$ and $Z = 1$. Equation (20) gives the generating functions of A_2- sequence as

$$A_2(t) = \left.\frac{h_1 h_2}{t^2}\right|_{t=t/\sqrt{1+t^2}} = 1 + t^2.$$

Sequence characterization of double Riordan array

Example 2.2 Kuznetkov, Pak, and Postnikov 1996 consider ordered trees with no points at odd heights, which is converted to the Dyck paths with no valley at odd heights by Davenport, Shapiro, and Woodson 2012. Denote by $d_{n,k}$ the number of those paths with n edges that end at height k. Then the matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 4 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & \dots \\ 4 & 0 & 5 & 0 & 3 & 0 & 1 & 0 & 0 & .. \\ 0 & 9 & 0 & 8 & 0 & 4 & 0 & 1 & 0 & \dots \\ 9 & 0 & 12 & 0 & 9 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{22}$$

Sequence characterization of double Riordan array

It can be seen that

$$\begin{aligned} d_{n,2k} &= d_{n-2,2k-2} + d_{n-2,2k} + d_{n-2,2k+2}, \quad k \geq 1, \\ d_{n,2k-1} &= d_{n-1,2k-2} + d_{n-1,2k}, \quad k \geq 1, \\ d_{n,0} &= d_{n-2,0} + d_{n-2,2} \end{aligned}$$

for $n \geq 1$. Thus, $A_1 = 1 + t^2$, $A_2 = 1 + t^2 + t^4$, and $Z = 1 + t^2$.

Sequence characterization of double Riordan array

Theorem

Let $(d; h_1, h_2)$ be a double Riordan array, and let $\overline{h_{1,2}}$ be the compositional inverse of $\sqrt{h_1 h_2}$, i.e.,
$(\overline{h_{1,2}} \circ \sqrt{h_1 h_2})(t) = (\sqrt{h_1 h_2} \circ \overline{h_{1,2}})(t) = t$. *Then it has a unique inverse*

$$(d; h_1, h_2)^{-1} = \left(\frac{1}{d(\overline{h_{1,2}})}, \overline{h_{1,2}} \frac{t}{h_1(\overline{h_{1,2}})}, \overline{h_{1,2}} \frac{t}{h_2(\overline{h_{1,2}})} \right). \qquad (23)$$

Particularly,

$$(d; t, h_2)^{-1} = \left(\frac{1}{d(\overline{h_{1,2}})}, t, \overline{h_{1,2}} \frac{t}{h_2(\overline{h_{1,2}})} \right), \qquad (24)$$

where $\overline{h_{1,2}}$ is the compositional inverse of th, i.e., $\overline{th_2}$.

Sequence characterization of double Riordan array

Theorem

Let A_1^*, A_2^*, and d^* be the generating functions of A_1-, A_2-, and Z-sequences of $(d; h_1, h_2)^{-1}$, where $(d; h_1, h_2) \in \mathcal{DS}$. Then we have

$$A_1^*\left(\frac{t\overline{h_{1,2}}}{\sqrt{h_1\left(\overline{h_{1,2}}\right) h_2\left(\overline{h_{1,2}}\right)}}\right) = \frac{1}{A_1(t)}, \tag{25}$$

$$A_2^*\left(\frac{t\overline{h_{1,2}}}{\sqrt{h_1\left(\overline{h_{1,2}}\right) h_2\left(\overline{h_{1,2}}\right)}}\right) = \frac{1}{A_2(t)}, \tag{26}$$

$$\begin{aligned} Z^*\left(\frac{t\overline{h_{1,2}}}{\sqrt{h_1\left(\overline{h_{1,2}}\right) h_2\left(\overline{h_{1,2}}\right)}}\right) &= \frac{1-d(\overline{h_{1,2}})}{t^2} \\ &= -\frac{\overline{h_{1,2}}^2}{t^2} d(\overline{h_{1,2}}) Z(t). \end{aligned} \tag{27}$$

Sequence characterization of double Riordan array

Example 2.3 $\hat{R}$ shown in (39) is the compression of the double Riordan array, the Pascal-Stanley array, $(1/(1-t^2), t, t/(1-t^2))$. The generating functions of its Z-, A_1-, and A_2- sequences are found in Example 2.1 as $A_1 = Z = 1$ and $A_2 = 1 + t^2$. Since $h_1 = t$ and $h_2 = t/(1-t^2)$, $\sqrt{h_1 h_2} = t/\sqrt{1-t^2}$. The compositional inverse of $\sqrt{h_1 h_2}$ is $\overline{h_{1,2}} = t/\sqrt{1+t^2}$. The inverse of $(1/(1-t^2), t, t/(1-t^2))$ is $(1/(1+t^2), t, t/(1+t^2))$. By using (19)-(21), we may find the generating functions of Z-, A_1-, and A_2sequences are

$$Z^* = -1, \quad A_1^* = 1, \quad A_2^* = 1 - t^2.$$

Noting

$$\frac{t\overline{h_{1,2}}}{h_1(\overline{h_{1,2}})h_2(\overline{h_{1,2}})} = \overline{h_{1,2}} = \frac{t}{\sqrt{1+t^2}},$$

we may check (27)-(29) hold.

The compressions of Double Riordan arrays

Let $(d; h_1, h_2) = (d_{n,k})_{n\geq k\geq 0}$ be a double Riordan array. We define its compression $(\hat{d}; \hat{h}_1, \hat{h}_2) = (\hat{d}_{n,k})_{n\geq k\geq 0}$ as forllows:

$$\hat{d}_{n,k} := d_{2n-k,k}, \quad n \geq k \geq 0, \tag{28}$$

or equivalently,

$$\hat{d}(t) = \sum_{k\geq 0} d_{2k} t^k, \ \hat{h}_1(t) = \sum_{k\geq 0} h_{1,2k+1} t^{k+1}, \text{ and } \hat{h}_2(t) = \sum_{k\geq 0} h_{2,2k+1} t^{k+1}. \tag{29}$$

As an example $\hat{R} = (1/(1-t), t, t/(1-t))$ shown in (39) is the compression of the double Riordan array $(1/(1-t^2), t, t/(1-t^2))$.

The compressions of Double Riordan arrays

Theorem

Let $(d; h_1, h_2) = (d_{n,k})_{n\geq k\geq 0}$ *be a double Riordan array, and let its compression be defined by* $(\hat{d}_{n,k})_{n\geq k\geq 0}$, *where* $\hat{d}_{n,k}$ *is shown in* (28). *Suppose the* A_1*-,* A_2*-, and* Z*-sequences of* $(d; h_1, h_2)$ *are*

$$A_1 = \{a_{1,0}, a_{1,1}, \ldots\}, \quad A_2 = \{a_{2,0}, a_{2,1}, \ldots\}, \quad \text{and} \quad Z = \{z_0, z_1, \ldots\}.$$

Then for $l = 1, 2, \ldots$ *and* $n = 1, 2, \ldots$,

$$\hat{d}_{n,2l-1} = a_{1,0}\hat{d}_{n-1,2l-2} + a_{1,1}\hat{d}_{n,2l} + a_{1,2}\hat{d}_{n+1,2l+2} + \cdots, \quad (30)$$

$$\hat{d}_{n,2l} = a_{1,0}\hat{d}_{n-2,2l-2} + a_{1,1}\hat{d}_{n-1,2l} + a_{1,2}\hat{d}_{n,2l+2} + \cdots, \quad (31)$$

$$\hat{d}_{n,0} = z_0\hat{d}_{n-1,0} + z_1\hat{d}_{n,2} + z_2\hat{d}_{n+1,4} + \cdots. \quad (32)$$

The compressions of Double Riordan arrays

Example 2.4 Let $D = (d_{n,k})_{n,k\geq 0}$ be the double Riordan array of the Dyck path numbers studied in Example 2.2. Then its compression is

$$\hat{D} = (\hat{d}_{n,k})_{n,k\geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 4 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 9 & 9 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 21 & 21 & 12 & 8 & 3 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{33}$$

It can be seen that

$$\begin{aligned} \hat{d}_{n,2l-1} &= \hat{d}_{n-1,2l-2} + \hat{d}_{n,2l}, \quad l \geq 1, \\ \hat{d}_{n,2l} &= d_{n-2,2l-2} + d_{n-1,2l} + \hat{d}_{n,2l+2}, \quad l \geq 1, \\ \hat{d}_{n,0} &= d_{n-1,0} + d_{n,2} \end{aligned}$$

for $n \geq 1$. Thus, $A_1 = 1 + t^2$, $A_2 = 1 + t^2 + t^4$, and $Z = 1 + t^2$.

The compressions of Double Riordan arrays

Theorem

$\mathcal{L} := \{(d; t, h) \in \mathcal{DR}\}$ is a subgroup of $\mathcal{DR}$. Furthermore, if $\{(d, h) \in \mathcal{R}\}$ is in Appel subgroup, associate subgroup, Bell subgroup of $\mathcal{R}$, then $\{(d; t, h) \in \mathcal{DR}\}$ is in Appel subgroup, associate subgroup, and type-2 Bell subgroup of $\mathcal{DR}$, respectively.

The compressions of high order Riordan arrays

Definition

Let $d(t) \in \mathcal{F}_0$ and $h_j(t) \in \mathcal{F}_1$, where $d(t) = \sum_{n\geq 0} d_{kn} t^{kn}$ and $h_j(t) = \sum_{n\geq 0} h_{j,kn+1} t^{kn+1}$, $1 \leq j \leq k$. Then the kth order Riordan array with respect to d and h_j $(1 \leq j \leq k)$ is defined by

$$\begin{aligned}(d; h_1, h_2, \ldots, h_k) &= (d, dh_1, dh_1h_2, \cdots d\Pi_{j=1}^{k} h_j, dh_1\Pi_{j=1}^{k} h_j, \\ &\quad dh_1h_2\Pi_{j=1}^{k} h_j, \cdots). \qquad (34)\end{aligned}$$

We denote the entry of nth row and jth column as $d_{n,j}$ and write $(d; h_1, h_2, \ldots, h_k) = (d_{n,j})_{n\geq j\geq 0}$.

The compressions of high order Riordan arrays

Theorem

Let $(d; h_1, h_2, \ldots, h_k)$ be a kth order Riordan array, and let $A_j(t) = t^j B_j(t)$, where $B_j(t) = \sum_{n\geq 0} b_{kn+j} t^{kn}$, $j = 0, 1, \ldots, k-1$. Then we have the following Fundamental Theorem of kth order Riordan Arrays (FTKRA):

$$
\begin{aligned}
& (d; h_1, h_2, \ldots, h_k) A_j(t) \\
= & \begin{cases} dB_0\left(\sqrt[k]{\Pi_{n=1}^{k} h_n(t)}\right) & \text{if } j = 0 \\ dh_1 h_2 \cdots h_j B_j\left(\sqrt[k]{\Pi_{n=1}^{k} h_n(t)}\right) & \text{if } 1 \leq j \leq k-1 \end{cases}
\end{aligned} \tag{35}
$$

The compressions of high order Riordan arrays

For kth order Riordan arrays $(d; h_1, h_2, \ldots, h_k)$ and $(g; f_1, f_2, \ldots, f_k)$,

$$\begin{aligned}&(d; h_1, h_2, \ldots, h_k) * (g; f_1, f_2, \ldots, f_k)\\ = &\left(dg\left(\sqrt[k]{\Pi_{n=1}^{k} h_n(t)}\right), \sqrt[k]{\frac{h_1^{k-1}}{h_2 h_3 \cdots h_k}} f_1\left(\sqrt[k]{\Pi_{n=1}^{k} h_n(t)}\right), \ldots, \right.\\ &\left. \sqrt[k]{\frac{h_k^{k-1}}{h_1 h_2 \cdots h_{k-1}}} f_k\left(\sqrt[k]{\Pi_{n=1}^{k} h_n(t)}\right)\right), \qquad (36)\end{aligned}$$

where $g(t) = \sum_{n\geq 0} g_{kn} t^{kn}$ and $f_j(t) = \sum_{n\geq 0} f_{j,kn+1} t^{kn+1}$, $1 \leq j \leq k$. The kth order Riordan arrays defined by (34) associated with the multiplication defined by (36) form a group, called kth order Riordan group. Particularly, for $k = 1$ and $k = 2$ the kth order Riordan group reduces to the Riordan group and double Riordan group, respectively.

The compressions of high order Riordan arrays

Definition

The compression of a kth order Riordan array shown in Definition 7 is defined by $(\hat{d}; \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_k) = (\hat{d}_{n,k})$ where

$$\hat{d}_{n,j} = d_{kn-(k-1)j,j}, \quad n \geq j \geq 0, \tag{37}$$

or equivalently,

$$\hat{d}(t) = \sum_{n\geq 0} d_{kn} t^n, \quad \hat{h}_j(t) = \sum_{n\geq 0} h_{j,kn+1} t^{n+1}.$$

The compressions of high order Riordan arrays

The third order Riordan array $(1/(1-t^3), t, t, t/(1-t^3))$ possesses its compression $(1/(1-t), t, t, t/(1-t))$, which first few rows can be presented as follows.

$$\bar{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .. \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 4 & 3 & 2 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 5 & 4 & 3 & 3 & 1 & 0 & \dots \\ 1 & 1 & 1 & 6 & 5 & 4 & 6 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \end{bmatrix} \tag{38}$$

Matrix $\bar{R}$ is called *Pascal-Narayana array*. The row sums of $\bar{R}$ are Narayana's cows numbers, $(a_n) = (1, 2, 3, 4, 6, 9, 13, 19, 28, \dots)$.

The compressions of high order Riordan arrays

The compressions of high order Riordan arrays are a type extensions of Riordan arrays. If $k = 1$, then the kth order Riordan array $(d; h_1, h_2, \ldots, h_k)$ is the Riordan array (d, h_1). In this sense, we may call the compression of a high order Riordan array a quasi-Riordan array. Conversely, each quasi-Riordan array can be considered as the compression of a high order Riordan array. There is a correspondence established via quasi-Riordan arrays between nth order Riordan array set and the set of $(n+1) \times (n+1)$ principal submatrices of lower triangular matrices.

The compressions of high order Riordan arrays

Let $(c_0, c_1, c_2, \ldots)$ be a lower triangular array with the zeroth, first, second, ... columns denoted by c_0, c_1, c_2, ..., respectively. Then its $(n+1) \times (n+1)$ principal submatrix can be written as $(c_0, c_1, c_2, \ldots, c_n)_{(n+1)\times(n+1)}$. We may extend $(c_0, c_1, c_2, \ldots, c_n)$ to a quasi-Riordan array $(\hat{d}; \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n)$, where $\hat{d}$ is the generating function of column c_0, $\hat{h}_1$ is the ratio of the generating functions of c_1 and c_0, $\hat{h}_2$ is the ratio of the generating functions of c_2 and c_1, ..., and $\hat{h}_n$ is the ratio of the generating functions of c_n and c_{n-1}. The columns after the first $n+1$ ones of the extension array of $(c_0, c_1, c_2, \ldots, c_n)$ follows the construction rule of quasi-Riodan arrays. The $n+2$ column of the extension array is $\hat{d}\hat{h}_1\Pi_{j=1}^{n}\hat{h}_j$, i.e., the multiplication of the generating functions of c_1 and c_n divided by the generating function of c_0, the generating function of the $n+3$ column of the extension array is $\hat{d}\hat{h}_1\hat{h}_2\Pi_{j=1}^{n}\hat{h}_j$, i.e., the multiplication of the generating functions of c_2 and c_n divided by the generating function of c_0, etc.

The compressions of high order Riordan arrays

A quasi-Riordan array can be considered as the compression of a nth order Riordan array $(d; h_1, h_2, \ldots, h_n)$, where d and h_j $(1 \geq j \geq n)$ are obtained as shown before from $\hat{d}$ and $\hat{h}_j$ $(1 \geq j \geq n)$, respectively. Hence, we have defined a mapping from the $(n+1) \times (n+1)$ principal submatrix of $(c_0, c_1, c_2, \ldots)$ to a nth order Riordan array. Conversely, a nth order (or infinite order) Riordan array can be compressed as a unique quasi-Riordan array, which is a lower triangular matrix associated to the nth order Riordan array. Hence, there is a correspondence between the set of $(n+1) \times (n+1)$ principal submatrices of lower triangular matrices and the set of nth order Riordan arrays.

The compressions of high order Riordan arrays

The first four columns of Narayana matrix $(N(n,k) = \binom{n}{k}\binom{n}{k-1}/n)$

$$\hat{N}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 6 & 6 & 1 \\ 1 & 10 & 20 & 10 \\ 1 & 15 & 50 & 50 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{1-t} & \frac{t}{(1-t)^3} & \frac{t^2(1+t)}{(1-t)^5} & \frac{t^4(1+3t+t^2)}{(1-t)^7} \end{bmatrix}. \tag{39}$$

$\hat{N}_4$ corresponds the 3rd Riordan array with the compression

$$(\hat{d}; \hat{h}_1, \hat{h}_2, \hat{h}_3) = \left(\frac{1}{1-t}; \frac{t}{(1-t)^2}, \frac{t(1+t)}{(1-t)^2}, \frac{t^2(1+3t+t^2)}{(1+t)(1-t)^2} \right).$$

The compressions of high order Riordan arrays

Can a low triangular matrix be decomposed into a linear combination of some multiplications of Riordan arrays and/or quasi-Riordan arras?

Stabilizer subgroups of $\mathcal{DR}$

A double Riordan array can be described using the evaluation at a series $f \in \mathcal{F}_0$ of the natural linear map induced by R in the linear space $K[[x]]$. So, essentially, what we do is to change the notation for Riordan matrices or, equivalently, they produce a change of variables in the double Riordan group. We now present another view to the FTRA by looking at the $(d; h_1, h_2)$ in the FTDRA (i.e., the FTRA for double Riordan arrays) as a transition operator acting on a set of generating functions of $\mathcal{F}_0$, or the double Riordan group acts on $\mathcal{F}_0$.

Stabilizer subgroups of $\mathcal{DR}$

Let G be a group, and let S be nonempty set. Then G is said to act on S if there is a function from $G \times S$ to S, i.e., $(g, s) \to gs$, such that $es = s$ and $(gh)s = g(hs)$ for all $g, h \in G$, and all $s = \sum_{k\geq 0} a_k t^k \in S$. Hence, we may say the double Riordan group $\mathcal{DR}$ act on $\mathcal{F}_0$ because of the following two reasons. First, $(1; t, t)s = s$, and second,

$$((d; h_1, h_2)(g; f_1, f_2))s = (d; h_1, h_2)((g; f_1, f_2)s)$$

for all $s \in \mathcal{F}_0$, and all $(d; h_1, h_2)$, $(g; f_1, f_2) \in \mathcal{DR}$. Particularly, if $(d; h_1, h_2)$ is an Appell, associated, and Bell type Riordan array, $(d; t, t)$, $(1; h_1, h_2)$, $(d; td, h_2)$, and $(d; h_1, td)$, respectively, then the corresponding $(d; h_1, h_2)$ are called the double Appell, associated, and Bell transition operators.

Stabilizer subgroups of $\mathcal{DR}$

If there exists a f.p.s $f \in \mathcal{F}_0$, i.e., a f.p.s f with $f(0) \neq 0$, such that $(d; h_1, h_2)f = f$ for all elements $(d; h_1, h_2)$ in a subgroup of the Riordan group, then we call f the stabilizer of the subgroup. For every $f \in \mathcal{F}_0$, we may define a relevant subgroup of the double Riordan group. More precisely, we have

$$\mathcal{DS}_f := \{(d; h_1, h_2) \in \mathcal{DR} : (d; h_1, h_2)f = f\}. \tag{40}$$

We call $\mathcal{S}_f$ the stabilizer subgroup associated with $f \in \mathcal{F}$.

Stabilizer subgroups of $\mathcal{DR}$

The stabilizer subgroup with the stabilizer $f = 1/(1-t)$ is called the stochastic subgroup because the row sums of each matrix in the subgroup can be considered as probabilities summating to 1. Choosing suitable f, we may characterize some groups of $\mathcal{R}$ by using their stabilizer property. The associate subgroup, $\{(1; h_1, h_2) \in \mathcal{DR} : h_1(0), h_2(0) = 0,\ h_1'(0),\ h_2'(0) \neq 0\}$, of the Riordan group $\mathcal{DR}$ is the stabilizer subgroup $\mathcal{DS}_1$ associated with 1, or equivalently, the stabilizer of the associated subgroup is $f = 1$.

Stabilizer subgroups of $\mathcal{DR}$

Jean-Louis and Nkwanta 2013 gives a expression of stabilizer subgroups in $\mathcal{R}$. Shapiro and H 2017 consider the FTRA as a group action on the set $\mathcal{F}_0$. Then any series in $\mathcal{F}_0$ defines a stabilizer subgroup. This gives us an infinite number of subgroups of the Riordan group a few of which are already well known. Some subgroups of $\mathcal{R}$ can be characterized by a function $f \in \mathcal{F}_0$ such as the stochastic subgroup $\mathcal{S}_{1/(1-t)}$ and associate subgroup $\mathcal{S}_1$, but not all the subgroups that we know. Cheon, Luzón, Morón, Prieto-Martinez, and Song in their paper study the stabilizer of a formal power series under the action induced by Riordan matrices. They get stabilizers in the fraction field using bi-infinite representations.

Stabilizer subgroups of $\mathcal{DR}$

Appell subgroup of $\mathcal{DR}$ has no stabilizer because for any $f \in \mathcal{F}_0$ we may write $f = f_e + f_o$, where $f_e = (f(t) + f(-t))/2$ is even and $f_o = (f(t) - f(-t))/2$ is odd. By acting (d, t, t) on f, we obtain

$$(d; t, t)f = (d; t, t)f_e + (d; t, t)f_o = df_e + df_o = df,$$

which implies $d = 1$. Hence, $\mathcal{S}_f$ is a different class of subgroups of $\mathcal{DR}$ determined by each $f \in \mathcal{F}_0$. In addition, there are two questions raised: First, which known subgroups have their stabilizers; Second, how to find the subgroups $\mathcal{S}_f$ for each $f \in \mathcal{F}_0$. We have done some investigation of the first question. The following proposition will demonstrate solutions of the second question.

Stabilizer subgroups of $\mathcal{DR}$

Proposition

Let $\mathcal{R}$ be the Riordan group, and let $\mathcal{S}_f$ be the stabilizer subgroup defined by (40) *associated with a function $f \in \mathcal{F}_0$. If $f = 1/(1-t)$, then $(d; h_1, h_2) \in \mathcal{S}_{1/(1-t)}$, where $h_1 = t$ and $h_2 \in \mathcal{F}_1$, if and only if*

$$d = \frac{1 - th_2}{1 - t^2} \tag{41}$$

with the constant term and the coefficient of the quadratic term of d as $d_0 = 1$ and $d_2 \neq 1$.

Stabilizer subgroups of $\mathcal{DR}$

Example 2.5 If $d = C^2(t^2)$, where C is the generating function of the Catalan numbers, then $d_0 = 1$ and $d_2 = 2$. To find $(C^2(t^2); h_1, h_2) \in S_{1/(1-t)}$, we use (41) to get $h_2 = (C(t^2) - C^2(t^2))/t$. Then from $C = 1 + tC^2$ we have

$$h_2 = \frac{C(t^2)(1 - C(t^2))}{t} = -tC^3(t^2).$$

Thus $(C^2(t^2); t, -tC^3(t^2)) \in \mathcal{S}_{1/(1-t)}$, i.e.,

$$(C^2(t^2); t, -tC^3(t^2))\frac{1}{1-t} = \frac{1}{1-t},$$

or equivalently

$$\frac{C^2(t^2)}{1 + tC^3(t^2)} = \frac{1}{1-t}.$$

From the proposition we know any double Riordan array $(d; t, h_2)$ with $d = 1/(1 - t^2)$ or $C(t^2)$ is not in the subgroup $\mathcal{S}_{1/(1-t)}$.

Stabilizer subgroups of $\mathcal{DR}$

Let $\mathcal{R}$ be the Riordan group, and let $\mathcal{S}_f$ be the stabilizer subgroup defined by (40) associated with a function $f \in \mathcal{F}_0$. If $f = 1/(1-kt)$, $k \geq 1$, then $(d; t, h_2) \in \mathcal{S}_{1/(1-kt)}$ if and only if even function $d \in \mathcal{F}_0$ with $d_0 = 1$ and $d_2 \neq k$ satisfy

$$d = \frac{1 - kth_2}{1 - k^2t^2}. \tag{42}$$

As an example we consider the case of $k = 2$. If $d = 1/(1-t^2)$, then $d_0 = d_2 = 1$. From (42), we have $h_2 = t/(2(1-t^2))$. Then for this h_2, $(1/(1-t^2); t, h_2) \in \mathcal{S}_{1/(1-2t)}$. If $d = C(t^2)$, then $d_0 = d_2 = 1$. From (42), we have

$$h_2 = \frac{1}{2t}(1 - C(t^2) + 4t^2C(t^2)) = \frac{tC(t^2)}{2}(4 - C(t^2))$$

and $(C(t^2); t, h_2) \in \mathcal{S}_{1/(1-2t)}$. However, for $d = C^2$ there does not exist an odd function $h_2 \in \mathcal{F}_1$ such that $(d; t, h_2) \in \mathcal{S}_{1/(1-2t)}$.

Stabilizer subgroups of $\mathcal{DR}$

Theorem

Let $(d; h_1, h_2) \in \mathcal{DR}$, and let $a \in \mathcal{F}_0$. Then the stabilizer group $\mathcal{DS}_a = \{(d; h_1, h_2) \in \mathcal{DR} : (d; h_1, h_2)a = a\}$ can be written as

$$\mathcal{DS}_a = \left\{ \left(\frac{2a\sqrt{h_2}}{a(\sqrt{h_1 h_2})(\sqrt{h_1} + \sqrt{h_2}) - a(-\sqrt{h_1 h_2})(\sqrt{h_1} - \sqrt{h_2})} ; h_1, h_2 \right) : h_1, h_2 \in \mathcal{F}_1 \text{are odd} \right\}. \tag{43}$$

Stabilizer subgroups of $\mathcal{DR}$

Theorem

Let $(d; h_1, h_2) \in \mathcal{DR}$ with $d(0) = 1$, and let $a \in \mathcal{F}_0$. Then $(d; h_1, h_2)$ is in the stabilizer group $\mathcal{DS}_f$ if and only if its A_1-, A_2-, and Z- functions satisfy

$$Z(t) = \frac{2a(\overline{h_{1,2}})A_2(t) - A_1(t)\sqrt{A_2(t)}(a(t) - a(-t)) - A_2(t)(a(t) + a(-t))}{2t^2 a(\overline{h_{1,2}})}. \tag{44}$$

Furthermore, if $(d; h_1, h_2) \in \mathcal{DR}$ has the A- and Z-characterizations satisfies (44) for some a, then $(d; h_1, h_2)$ is an element of the stabilizer subgroup $\in \mathcal{S}_a$.

Stabilizer subgroups of $\mathcal{DR}$

Let $(d; h_1, h_2) \in \mathcal{DR}$ with $h_1 = h_2$. Then $\overline{h_{1.2}} = \bar{h}_1$, where $\bar{h}_1$ is the compositional inverse of h_1, i.e., $h_1(\bar{h}_1) = \bar{h}_1(h_1) = t$. The A-characterizations are $A_1 = t/\bar{h}_1$ and $A_2 = A_1^2$. Then $(d; h_1, h_2) \in \mathcal{S}_a$ if $d = a/a(h_1)$, and the Z-characterization is

$$Z = A_1^2 \frac{a(\bar{h}_1) - a(t)}{t^2 a(\bar{h}_1)}.$$

Particularly, if $h_1 = h_2 = t$, then $A_1 = A_2 = 1$ and $Z = 0$, and $(d; h_1, h_2) = (1, t)$ is the identity Riordan array.
The double Riordan array $(1/(1-t^2); t, t/(1-t^2))$ has the Z-, A_1-, and A_2- functions, respectively,

$$Z(t) = 1, \quad A_1(t) = 1, \quad \text{and} \quad A_2(t) = 1 + t^2.$$

The compositional inverse of $\sqrt{h_1 h_2} = t/\sqrt{1-t^2}$ is $\overline{h_{1,2}} = t/\sqrt{1+t^2}$. Substituting $Z = 1$, $A_1 = 1$, $A_2 = 1 + t^2$, and $\overline{h_{1,2}} = t/\sqrt{1+t^2}$ into (44), we may obtain $a = 1/t^2$., namely, the Pascal-Fibonacci array $(1/(1-t^2); t, t/(1-t^2)) \in \mathcal{DS}_{1/t^2}$.

P matrices of double Riordan arrays

Let $(d; h_1, h_2)$ be a double Riordan array, and let $\{z_n\}_{n\geq 0}$, $\{a_{1n}\}_{n\geq 0}$, and $\{a_{2n}\}_{n\geq 0}$ are the Z-, A_1-, and A_2- sequences of the double Riordan array. Define

$$P = \begin{bmatrix} z_0 & a_{10} & a_{20} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ z_1 & a_{11} & a_{21} & a_{10} & a_{20} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ z_2 & a_{12} & a_{22} & a_{11} & a_{21} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and call it the P matrix of $(d; h_1, h_2)$. It can be seen that

$$(d; h_1, h_2)P = \Big(dZ(\sqrt{h_1 h_2}), dA_1(\sqrt{h_1 h_2}),$$

$$dA_2(\sqrt{h_1 h_2}), dh_1 h_2 A_1(\sqrt{h_1 h_2}), dh_1 h_2 A_2(\sqrt{h_1 h_2}), \dots \Big).$$

P matrices of double Riordan arrays

Define the shift matrix U by

$$U = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We have

$$(d; h_1, h_2)P = U(d; h_1, h_2).$$

Applications include the decomposition of double Riordan arrays in terms of P (see H LAA 2015 for RA), matrix characterization of double Riordan arrays, and identities (see H LAA 2016 for RA).

Sheffer sequences associated with double Riordan arrays

Let $(g; f_1, f_2)$ be a double Riordan array, and let its inverse is

$$(d; h_1, h_2) = (g; f_1, f_2)^{-1} = \left(\frac{1}{g(\overline{f_{1,2}})}; \overline{f_{1,2}} \frac{t}{f_1(\overline{f_{1,2}})}, \overline{f_{1,2}} \frac{t}{f_2(\overline{f_{1,2}})} \right), \quad (45)$$

where $\overline{f_{1,2}}$ is the compositional inverse of $\sqrt{f_1 f_2}$. Let $d(t) \in \mathcal{F}_0$ and $h_1(t), h_2(t) \in \mathcal{F}_1$; we say that the sequence $(s_{1,n}(x), s_{2,n})_{n\in\mathbb{N}}$ is *Sheffer pair* for the triple $(d(t), h_1(t), h_2(t))$ if and only if

$$\sum_{k\geq 0} s_{1,k}(x) \frac{t^{2k}}{c_{2k}} = \frac{1}{d} \varepsilon_x^{(1)}(h_1(t)h_2(t)) \quad \text{and}$$

$$\sum_{k\geq 0} s_{2,k}(x) \frac{t^{2k+1}}{c_{2k+1}} = \frac{1}{d} \varepsilon_x^{(2)}(h_1(t)h_2(t)),$$

where $\varepsilon_x^{(1)}(uv) = \sum_{k\geq 0} x^{2k}(uv)^k/c_{2k}$ and $\varepsilon_x^{(2)}(uv) = u\sum_{k\geq 0} x^{2k+1}(uv)^k/c_{2k+1}$.

Sheffer sequences associated with double Riordan arrays

If the triple (d, h_1, h_2) form a double Riordan array $(d; h_1, h_2)$, then from the above definition of Sheffer pair, we have

$$s_{1,n}(x) = \sum_{k\geq 0} d_{2n,2k}\frac{x^{2k}}{c_{2k}} \quad \text{and} \quad s_{2,n}(x) = \sum_{k\geq 0} d_{2n+1,2k+1}\frac{x^{2k+1}}{c_{2k+1}}, \tag{46}$$

where $d_{n,2k} = [t^n]d(h_1h_2)^k$ and $d_{n,2k+1} = [t^n]dh_1(h_1h_2)^k$.

Thank you!