Fuzzy Sets and Systems 161 (2010) 412-432
Elsevier www.elsevier.com/locate/fss

A Categorical Semantics for Fuzzy Predicate Logic

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Abstract

The object of this study is to look at categorical approaches to many valued logic, both propositional and predicate, to see how different logical properties result from different parts of the situation. In particular, the relationship between the *categorical. fabric I* introduced at Linz in 2004 and the Fuzzy Logics studied by Hajek (2003) [5], Esteva et al. (2003) [1], and Hajek (1998) [4], comes from restricting the kind of structures used for truth values. We see how the structure of the various kinds of algebras shows up in the categorical logic, giving a variant on natural deduction for these logics. Quantification typically needs more completeness than is present in the algebras used in Hajek (1998) [4], hence the need for *safe interpretations*. The categorical setting gives a predicate logic without variables. The language in the more traditional sense comes from a structure built on a particular freely generated Cartesian category. Formulas have a clear meaning in that more restricted context. Interpretation of the language in other categorical fabrics is given by application of a product preserving functor. Traditional completeness results relate to this kind of interpretation. Completeness can also be understood as showing that the derivable truths in the general fabric are the necessary truths: those which are true in all of the possible worlds.

1. Categorical propositional logic

The use of category theory to investigate logical structures has a fairly long history with classic texts by Lambek and Scott **1101** and Makkai and Reyes 1121. To apply the approach to fuzzy logic we want a setting with a monoidal structure and we may want to work either internally as in **1181** (which aimed at an internal higher order fuzzy logic) or in a more external form which is closer to common practice. In addition we may want inferences for DeMorgan negations and for the A operator used in some of the fuzzy literature.

The categorical setting of this paper assumes that we have an underlying category T of types. We assign to each object A of T a category P(A) of predicates about A. The categories P(A) have specified additional structures (things like limits, colimits, monoidal structures). This assignment is a contravariant functor from T into the category of categories with the specified structures taking each morphism $f : A \to \boldsymbol{B}$ a functor $f^* : P(B) \to$ P(A) which preserves the specified additional structure. We use the notation $0\mathrm{I}\ \mathrm{F}_{\mathrm{A}}\ {}^{I}\!I^{i}$ to indicate that there is a morphism from c to J*i* in P(A).

This says somewhat more than the semantic entailment given by $\phi \models \psi$ which is taken to mean that whenever ϕ is fully true, ψ will be also. In a fuzzy setting we want to say that ψ is at least as true as ϕ, though neither may be fully true. In most cases we do not keep track of whether there are more than one morphisms from ϕ to ψ: presumably we could decorate the notation appropriately if we wanted to.

Example (*Categorical propositional logics*). In a topos $\mathcal{E}$ the propositional logic sits over $\mathcal{E}$ as category of types with $\mathcal{P}(A)$ given by the category of subobjects of A. In a quasitopos, $\mathcal{P}(A)$ would be the lattice of strong subobjects.

In fuzzy set theory we often take the category of types to be **Sets** and let $\mathcal{P}(A)$ be the lattice of fuzzy subsets of A. In the setting of [18] the types would be the whole Goguen category **Set**(L) and the categories of predicates about a fuzzy set (A, α) would be the lattice of unbalanced subobjects $\mathcal{U}(A, \alpha)$.

Höhle's construction in [7] based on the Higgs topos [6] has types given by sets with an L-valued similarity relation and uses the lattice of t-tight subobjects.

1.1. Logical structures inherent in categories

If all we know about the categories of predicates is that they are categories then what we get are the axioms

$$\alpha \Vdash_A \alpha$$

from the identity maps and the rule

$$\frac{\alpha \Vdash_A \beta \quad \beta \Vdash_A \gamma}{\alpha \Vdash_A \gamma}$$

from the composition. The equations in categories giving associativity of composition and the fact that identity morphisms are identities for composition provide the means for determining equivalent proofs.

Any covariant functor $F : \mathcal{P}(B) \to \mathcal{P}(A)$ will induce a rule

$$\frac{\phi \Vdash_B \psi}{F(\phi) \Vdash_A F(\psi)} \text{F-functor}$$

In particular, for any morphism $f : A \to B$ we get a rule

$$\frac{\phi \Vdash_B \psi}{f^\star(\phi) \Vdash_A f^\star(\psi)} f^\star\text{-functor}$$

If we have a pair of adjoint functors $G \dashv F$ where $F : \mathcal{P}(A) \to \mathcal{P}(B)$ and $G : \mathcal{P}(B) \to \mathcal{P}(A)$ then we get a rule

$$\frac{\phi \Vdash_B F(\xi)}{\overline{G(\phi) \Vdash_A \xi}} G \dashv F$$

This will be used to get the rules for implication and for quantification. The double line means that we have both of the inferences

$$\frac{\phi \Vdash_B F(\xi)}{G(\phi) \Vdash_A \xi} \quad \frac{G(\phi) \Vdash_A \xi}{\phi \Vdash_B F(\xi)}$$

Any contravariant functor $G : \mathcal{P}(B) \to \mathcal{P}(A)$ will induce a rule

$$\frac{\phi \Vdash_B \psi}{G(\psi) \Vdash_A G(\phi)} \text{G-contrafunctor}$$

This arises in fuzzy set theory when we posit a negation which is an idempotent involution.

Additional structures assumed for the categories $\mathcal{P}(A)$ will give us additional rules:

- *Products*: If the category $\mathcal{P}(A)$ is assumed to have pairwise products, then the projections $\pi_1 : \phi \wedge \psi \to \phi$ and $\pi_2 : \phi \wedge \psi \to \psi$ give rules:

$$\frac{\xi \Vdash_A \phi \wedge \psi}{\xi \Vdash_A \phi} \wedge\text{-EL} \quad \text{and} \quad \frac{\xi \Vdash_A \phi \wedge \psi}{\xi \Vdash_A \psi} \wedge\text{-ER}$$

  and the universal mapping property gives

$$\frac{\phi \Vdash_A \xi \quad \psi \Vdash_A \xi}{\phi \wedge \psi \Vdash_A \xi} \wedge\text{-I}$$

- *Coproducts*: If the category $\mathcal{P}(A)$ is assumed to have pairwise coproducts, then the injections $\iota_1 : \phi \to \phi \vee \psi$ and $\iota_2 : \psi \to \psi \vee \psi$ give rules:

$$\frac{\xi \Vdash_A \phi}{\xi \Vdash_A \phi \vee \psi} \vee\text{-IL} \quad \text{and} \quad \frac{\xi \Vdash_A \psi}{\xi \Vdash_A \phi \vee \psi} \vee\text{-IR}$$

  and the universal mapping property gives

$$\frac{\phi \Vdash_A \xi \quad \psi \Vdash_A \xi}{\phi \vee \psi \Vdash_A \xi} \vee\text{-E}$$

  Proof by cases comes from

$$\frac{\chi \Vdash_A \phi \vee \psi \quad \dfrac{\phi \Vdash_A \xi \quad \psi \Vdash_A \xi}{\phi \vee \psi \Vdash_A \xi} \vee\text{-E}}{\chi \Vdash_A \xi}$$

- *Terminal*: A terminal object $\top$ will give a rule

$$\frac{\top \Vdash_A \phi}{\xi \Vdash_A \phi} \top$$

  which gives an easy proof of $\phi \Vdash_A \top$.
- *Initial*: An initial object $\bot$ gives

$$\frac{\psi \Vdash_A \bot}{\psi \Vdash_A \xi} \bot$$

  which in turn gives an easy proof of $\bot \Vdash_A \xi$.
- *Limits and colimits in general*: Larger limit diagrams and colimit diagrams will give infinitary inference rules. For example if we use a diagram

$$\mathbb{N}_{\to} = 0 \to 1 \to 2 \to \dots$$

  we can get infinitary inferences

$$\frac{\phi_0 \Vdash_A \psi, \ \phi_1 \Vdash_A \psi, \ \dots, \ \{\phi_i \Vdash_A \phi_{i+1}\}_{i \in \mathbb{N}}}{\lim_{\to} \phi_i \Vdash \phi}$$

  from the existence of the colimit and

$$\frac{\phi \Vdash_A \psi_0, \ \phi \Vdash_A \psi_1, \ \dots, \ \{\psi_i \Vdash_A \psi_{i+1}\}_{i \in \mathbb{N}}}{\phi \Vdash \lim_{\leftarrow} \phi_i}$$

  from the existence of the limit.

The inferences for large products and coproducts are somewhat simpler:

$$\frac{\phi_0 \Vdash_A \psi,\ \phi_1 \Vdash_A \psi \ \ldots}{\bigvee_{i\in\mathbb{N}} \phi_i \Vdash_A \psi} \quad \text{and} \quad \frac{\phi \Vdash_A \psi_0,\ \phi \Vdash_A \psi_1,\ \ldots}{\phi \Vdash \bigwedge_{i\in\mathbb{N}} \psi_i}$$

- *Cartesian closed structure*: If each of the $\mathcal{P}(A)$ is Cartesian closed, then the functor $- \wedge \phi$ has a right adjoint $\phi \Rightarrow -$ and we get the rules

$$\frac{\psi \Vdash_A \phi \Rightarrow \xi}{\phi \wedge \psi \Vdash_A \xi} \text{ Cartesian}$$

Since both $- \wedge \phi$ and $\phi \Rightarrow -$ are covariant functors we also get inferences

$$\frac{\psi \Vdash_A \xi}{\psi \wedge \phi \Vdash_A \xi \wedge \phi} \quad \text{and} \quad \frac{\psi \Vdash_A \xi}{\phi \Rightarrow \psi \Vdash_A \phi \Rightarrow \xi}$$

- *Monoidal closed structure*: A monoidal structure $\phi \& \psi$ which is closed with right adjoint to $- \& \phi$ given by $\phi \to -$ gives a rule

$$\frac{\xi \Vdash_A \phi \to \psi}{\xi \& \phi \Vdash_A \psi} \text{ closed}$$

The monoidal structure calls for natural transformations $r_\phi : \phi \& I \to \phi$ and $l_\phi : I \& \phi \to \phi$ and $\alpha_{\phi\psi\xi} : (\phi \& \psi) \& \xi \to \phi \& (\psi \& \xi)$ which give rise to axioms allowing us to replace $I \& \phi \Vdash \phi$, $\phi \& I \Vdash \phi$, and $(\phi \& \psi) \& \xi \Vdash \phi \& (\psi \& \xi)$ together with the converses $\phi \Vdash I \& \phi$, $\phi \Vdash \phi \& I$, and $\phi (\& \psi \& \xi) \Vdash (\phi \& \psi) \& \xi$. If the monoidal structure is symmetric we get a natural isomorphism giving $\phi \& \psi \Vdash \psi \& \phi$. Since both $- \& \phi$ and $\phi \to -$ are covariant functors we also get inferences

$$\frac{\psi \Vdash_A \xi}{\psi \& \phi \Vdash_A \xi \& \phi} \quad \text{and} \quad \frac{\psi \Vdash_A \xi}{\phi \to \psi \Vdash_A \phi \to \xi}$$

- *DeMorgan negation*: If each $\mathcal{P}(A)$ is equipped with a DeMorgan negation given by a self-adjoint, contravariant, idempotent involution, then we get rules:

$$\frac{\phi \Vdash_A \psi}{-\psi \Vdash_A -\phi} \quad \text{and} \quad \frac{\phi \Vdash -\psi}{\psi \Vdash -\phi}$$

1.3. Two examples: fuzzy propositional logic arising from a t-norm

When the category of types is **Sets**:

- the terminal $T = \{*\}$ is a generator: a set S is determined by its elements, $s \in S$, which are the same as sections $\ulcorner s \urcorner : T \to S$ with $\ulcorner s \urcorner(*) = s$ and distinct functions are distinguishable by sections.
- if the entailment relation in each $\mathcal{P}(S)$ is determined by elementwise evaluation, that is, $\phi \Vdash_S \xi$ if and only if for every $\ulcorner s \urcorner : T \to S$ we have $\ulcorner s \urcorner^\star \phi \Vdash_T \ulcorner s \urcorner^\star \xi$ then the logic is truth functional.
- In this case $\mathcal{P}(T)$ is the category of truth values.

The categorical semantics of fuzzy predicate logic will be truth functional. The category of truth values is given by the unit interval with the order giving morphisms and a continuous t-norm & giving rise to a monoidal structure. Recall that a t-norm is a binary operation on the unit interval which preserves order and is associative, commutative, and has 1 as a unit. Continuity gives distributivity of & over large sups and thus gives a residuation. We use this structure to construct a setting for fuzzy propositional logic.

Definition 1. The **standard** categorical semantics for fuzzy sets with values in the interval with the t-norm & has **Sets** as the category of types and as $\mathcal{P}(A)$ the partially ordered set of all [0,1]-valued fuzzy sets $\alpha : A \to [0, 1]$. Here $\alpha \leq \beta$ if and only if $\forall_{a\in A}(\alpha(a) \leq \beta(a))$. A function $f : A \to B$ gives rise to a functor $f^\star : \mathcal{P}(B) \to \mathcal{P}(A)$ taking $\beta : B \to [0, 1]$ to $\beta \circ f : A \to B \to [0, 1]$.

This gives a truth functional logic in which all of the categories of predicates are symmetric monoidal closed, complete, cocomplete, and have the top element as the unit for the monoidal structure. The $f^\star$ functors preserve all of the structure. The categories $\mathcal{P}(A)$ should all satisfy the axioms of Hájek's BL [4]:

Because the top element is the unit for the monoidal structure in order to prove an implication $\phi \to \psi$ it suffices to give a proof of $\phi \Vdash \psi$:

$$\frac{\top \& \phi = \phi \Vdash \psi}{\top \Vdash \phi \to \psi}$$

To get the axioms for Hájek's BL we look at each in turn:

- A1: $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$. This follows from the associativity of & and the fact that $\top$ is an identity for &:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \to \psi \Vdash \phi \to \psi}{\phi \,\&\, (\phi \to \psi) \Vdash \psi}}{(\phi \,\&\, (\phi \to \psi)) \,\&\, (\psi \to \chi) \Vdash \psi \,\&\, (\psi \to \chi)} \quad \dfrac{\psi \to \chi \Vdash \psi \to \chi}{\psi \,\&\, (\psi \to \chi) \Vdash \chi}}{(\phi \,\&\, (\phi \to \psi)) \,\&\, (\psi \to \chi) = \phi \,\&\, ((\phi \to \psi) \,\&\, (\psi \to \chi)) \Vdash \chi}}{(\phi \to \psi) \,\&\, (\psi \to \chi) \Vdash (\phi \to \chi)}}{\top \,\&\, (\phi \to \psi) = (\phi \to \psi) \Vdash (\psi \to \chi) \to (\phi \to \chi)}}{\top \Vdash (\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))}$$

- A2: $(\phi \& \psi) \to \phi$. Follows from $\phi \&-$ being a functor and the fact that the top of the lattice is a unit for &:

$$\dfrac{\dfrac{\dfrac{\top \Vdash \top}{\psi \Vdash \top}\ \top}{\top \& (\phi \& \psi) = (\phi \& \psi) \Vdash \phi = \phi \& \top}\ \phi \&- \text{ functor}}{\top \Vdash (\phi \& \psi) \to \phi}$$

- A3: $(\phi \& \psi) \to (\psi \& \phi)$. This follows from an axiom that

  $$\phi \& \psi \Vdash \psi \& \phi$$

  which is natural since commutativity of the t-norm is an axiom and it gives rise to commutativity of the monoidal structure on the categories of predicates.
- A4: $(\phi \& (\phi \to \psi)) \to (\psi \& (\psi \to \phi))$. This says that product is definable in terms of tensor: it is special. It will be provable from the axiom $\phi \wedge \psi = \phi \& (\phi \to \psi)$ which makes that definition explicit.
- A5a: $(\phi \to (\psi \to \chi)) \to ((\phi \& \psi) \to \chi)$. This comes from commutativity, associativity, and residuation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \to (\psi \to \chi) \Vdash \phi \to (\psi \to \chi)}{\phi \& (\phi \to (\psi \to \chi)) \Vdash \psi \to \chi}}{\psi \& (\phi \& (\phi \to (\psi \to \chi))) \Vdash \psi \& (\psi \to \chi)} \quad \dfrac{\psi \to \chi \Vdash \psi \to \chi}{\psi \& (\psi \to \chi) \Vdash \chi}}{(\phi \& \psi) \& (\phi \to (\psi \to \chi)) \Vdash \psi \& (\psi \to \chi)}}{(\phi \& \psi) \& (\phi \to (\psi \to \chi)) \Vdash \chi}}{(\phi \to (\psi \to \chi)) \Vdash ((\phi \& \psi) \to \chi)}$$

- A5b: $((\phi \& \psi) \to \chi) \to (\phi \to (\psi \to \chi))$. This also comes from commutativity, associativity, and residuation:

$$\dfrac{\dfrac{\dfrac{(\phi \& \psi) \to \chi \Vdash (\phi \& \psi) \to \chi}{(\phi \& \psi) \& ((\phi \& \psi) \to \chi) = \psi \& (\phi \& ((\phi \& \psi) \to \chi)) \Vdash \chi}}{\phi \& (\phi \& \psi) \to \chi \Vdash \psi \to \chi}}{(\phi \& \psi) \to \chi \Vdash \phi \to (\psi \to \chi)}$$

- A6: $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$. Comes from quasilinearity, $\top \Vdash (\phi \to \psi) \vee (\psi \to \phi)$, using a proof by cases argument:

$$\dfrac{\top \Vdash (\phi \to \psi) \vee (\psi \to \phi) \quad \dfrac{B}{A} \quad \dfrac{D}{C}}{\top \Vdash ((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)}$$

$\frac{B}{A}$ where is

$$\dfrac{\dfrac{(\phi \to \psi) \to \chi \Vdash (\phi \to \psi) \to \chi \quad \chi \& ((\psi \to \phi) \to \chi) \Vdash \chi \& \top = \chi}{(\phi \to \psi) \& ((\phi \to \psi) \to \chi) \Vdash \chi} \quad \chi \Vdash ((\psi \to \phi) \to \chi) \to \chi}{(\phi \to \psi) \& ((\phi \to \psi) \to \chi) \Vdash ((\psi \to \phi) \to \chi) \to \chi}$$
$$(\phi \to \psi) \Vdash ((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$$

and we let

$$E = (\psi \to \phi) \& ((\psi \to \phi) \to \chi)$$

$\frac{D}{C}$ and then is

$$\dfrac{\dfrac{(\phi \to \psi) \to \chi \Vdash \top \quad (\psi \to \phi) \to \chi \Vdash (\psi \to \phi) \to \chi}{((\phi \to \psi) \to \chi) \& E \Vdash E} \quad E \Vdash \chi}{(\psi \to \phi) \& ((\phi \to \psi) \to \chi) \& ((\psi \to \phi) \to \chi) \Vdash \chi}$$
$$\dfrac{(\psi \to \phi) \& ((\phi \to \psi) \to \chi) \Vdash ((\psi \to \phi) \to \chi) \to \chi}{(\psi \to \phi) \Vdash ((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)}$$

Quasilinearity gets truth value 1 for any t-norm since the unit interval is linearly ordered. Truth functionality then gives $\top \Vdash_S (\phi \to \psi) \vee (\psi \to \phi)$ since for any $\ulcorner s \urcorner$ we get

$$\top \Vdash_T \ulcorner s \urcorner^\star ((\phi \to \psi) \vee (\psi \to \phi)) = (\ulcorner s \urcorner^\star (\phi) \to \ulcorner s \urcorner^\star (\psi)) \vee (\ulcorner s \urcorner^\star (\psi) \to \ulcorner s \urcorner^\star (\phi))$$

- A7: $\bot \to \phi$ follows from initial object inferences.

A second approach to logic of fuzzy sets is to take the internal logic of unbalanced subobjects as in [18]. Here the underlying category of types is the Goguen category **Set(I)** and the category of predicates about a fuzzy set (A, α) is given by the subcategory $\mathcal{U}(A)$ of **Set(I)**/(A, α) consisting of morphisms which are both monic and epic. These are morphisms $(A, \alpha') \to (A, \alpha)$ not changing the set involved but with $\alpha'(a) \leq \alpha(a)$ for all $a \in A$. In this case the underlying category of types does not have the terminal as a generator, one needs all of the unbalanced subobjects of the terminal to get a generating set (hence the information one gets from level sets). While each of the categories $\mathcal{P}(A, \alpha)$ has a terminal, it is not the unit for a monoidal structure. Indeed, the t-norm gives a commutative, associative operation with an adjoint implication, but in general it does not have a unit. This means that the logic of unbalanced subobjects is captured in inference statements but not necessarily in axioms using implication operators. The subobject categories $\mathcal{P}(A, \alpha)$ are still Cartesian closed (the top element is still a unit for $\wedge$) so the intuitionistic part of fuzzy logic still has more conventional form.

2. Predicate logic in a categorical setting

The logic of the individual categories $\mathcal{P}(A)$ is a categorical form of propositional logic. To get quantification we need to look at change of type.

2.1. Change of type and quantification as adjoint

Lawvere [11] noticed that quantification could be described in terms of adjoint functors. If our categories of predicates have enough completeness and cocompleteness then the change of type functors $f^\star$ will have adjoints giving quantifiers. (Hájek's safe interpretations restrict to those situations where the adjoints exist—this always happens when the fibers are finite, but may also happen in more general situations.)

If we restrict our attention to fuzzy predicate logic over **Sets** with values in a particular complete lattice L for each set S we get a category $\mathcal{P}_L(S)$ (typically a partial order) of predicates about S, each identified with a truth function

$\sigma : S \to L$, an L-fuzzy subset of S. The order (and much other structure) is inherited from that on L. These categories of predicates are connected to each other using trios of functors:

Theorem 1. *For any function $f : S \to T$ there are functors $f^\star : \mathcal{P}_L(T) \to \mathcal{P}_L(S)$, $\exists_f : \mathcal{P}_L(S) \to \mathcal{P}_L(T)$ and $\forall_f : \mathcal{P}_L(S) \to \mathcal{P}_L(T)$ with $\exists_f \dashv f^\star \dashv \forall_f$. Furthermore, a pullback square in* ***Sets***

$$\begin{array}{rcl} S & \xrightarrow{f} & T \\ h \downarrow & \textit{pull} & \downarrow g \\ U & \xrightarrow{k} & V \end{array}$$

gives rise to the Beck conditions

$$\exists_h f^\star = k^\star \exists_g \quad \textit{and} \quad \forall_h f^\star = k^\star \forall_g$$

as in the internal logic of topoi.

Proof. Given f we define the functors as follows:

$$f^\star(T, \tau)(s) = \tau(f(s))$$
$$\exists_f(S, \sigma)(t) = \bigvee \{\sigma(s) | f(s) = t\}$$
$$\forall_f(S, \sigma)(t) = \bigwedge \{\sigma(s) | f(s) = t\}$$

If f is the identity all of these are the identity functor. With these definitions the adjointness relations come from calculations:

$$f^\star \exists_f(S, \sigma)(s) = \bigvee \{\sigma(s') | f(s) = f(s')\} \geq \sigma(s)$$
$$\exists_f f^\star(T, \tau)(t) = \bigvee \{\tau(f(s)) | f(s) = t\} \leq \tau(t)$$
$$f^\star \forall_f(S, \sigma)(s) = \bigwedge \{\sigma(s') | f(s') = f(s)\} \leq \sigma(s)$$
$$\forall_f f^\star(T, \tau)(t) = \bigwedge \{\tau(f(s)) | f(s) = t\} \geq \tau(t)$$

where for both $\forall_f f^\star$ and $\exists_f f^\star$ we get equality if $\{\tau(f(s)) | f(s) = t\} \neq \emptyset$.

Similarly, given a pullback square

$$\begin{array}{rcl} S & \xrightarrow{f} & T \\ h \downarrow & \text{pull} & \downarrow g \\ U & \xrightarrow{k} & V \end{array}$$

we get the Beck conditions from:

$$\exists_h f^\star(T, \tau)(u) = \bigvee \{\tau(f(s)) | h(s) = u\}$$
$$k^\star \exists_g(T, \tau)(u) = \bigvee \{\tau(t') | g(t') = k(u)\}$$
$$\forall_h f^\star(T, \tau)(u) = \bigwedge \{\tau(f(s)) | h(s) = u\}$$
$$k^\star \forall_g(T, \tau)(u) = \bigwedge \{\tau(t') | g(t') = k(u)\}$$

Now the pullback S can be thought of as consisting of pairs $s = (u, t)$ such that $k(u) = g(t)$. We then get $h(s) = u$ and $f(s) = t$. This means that $\{\tau(f(s))|h(s) = u\} = \{\tau(t')|g(t') = k(u)\}$ since $\{f(s)|h(s) = u\} = \{t'|g(t') = k(u)\}$. $\square$

Since $\exists_f$, $\forall_f$ and $f^\star$ are functors we get rules of inference

$$\frac{\phi \Vdash_A \psi}{\exists_f(\phi) \Vdash_B \exists_f(\psi)} \quad \frac{\phi \Vdash_A \psi}{\forall_f(\phi) \Vdash_B \forall_f(\psi)} \quad \frac{\phi \Vdash_B \psi}{f^\star(\phi) \Vdash_A f^\star(\psi)}$$

The adjointness gives rules of inference:

$$\frac{\exists_f \phi \Vdash_B \psi}{\phi \Vdash_A f^\star(\psi)} \; \exists_f \dashv f^\star$$

$$\frac{\psi \Vdash_B \forall_f \phi}{f^\star(\psi) \Vdash_A \phi} \; f^\star \dashv \forall_f$$

where the double line indicates a reversible inference, giving both rules for introduction and elimination of quantifiers.

Since these are not the usual rules of inference for predicate logic it pays to see how they lead to both the usual axioms and to the generalization rule. One complication is that there are no variables and no constants, indeed no terms at all in the presentation given above for predicate logic: we transport predicates using $f^\star$ (which has the effect of adding free variables when applied to projection maps), $\exists_f$ and $\forall_f$ (which give usual quantification when applied to the unique map to the terminal type T). What we would usually write as $\phi(a)$ is $\ulcorner a \urcorner^\star \phi$, a predicate of type T where $\ulcorner a \urcorner : T \to A$ names the element $a \in A$: constants are global sections. Such a section will have the property that

$$T \xrightarrow{\ulcorner \bullet \urcorner} A \xrightarrow{!} T = \mathrm{id}_T : T \to T$$

Now both $\forall_{\mathrm{id}}$ and $\exists_{\mathrm{id}}$ are the identity functors so

$$\frac{\dfrac{\ulcorner a \urcorner^\star \phi \Vdash_T \ulcorner a \urcorner^\star \phi}{\phi \Vdash_A \forall_{\ulcorner \bullet \urcorner} \ulcorner a \urcorner^\star \phi} \; \ulcorner a \urcorner^\star \dashv \forall_{\ulcorner \bullet \urcorner}}{\forall_! \phi \Vdash_T \forall_! \forall_{\ulcorner \bullet \urcorner} \ulcorner a \urcorner^\star \phi = \ulcorner a \urcorner^\star \phi} \; \forall_! \text{ functor}$$

and

$$\frac{\dfrac{\ulcorner a \urcorner^\star \phi \Vdash_T \ulcorner a \urcorner^\star \phi}{\exists_{\ulcorner \bullet \urcorner} \ulcorner a \urcorner^\star \phi \Vdash_A \phi} \; \exists_{\ulcorner \bullet \urcorner} \dashv \ulcorner a \urcorner^\star}{\exists_! \exists_{\ulcorner \bullet \urcorner} \ulcorner a \urcorner^\star \phi = \ulcorner a \urcorner^\star \phi \Vdash_T \exists_! \phi} \; \exists_! \text{ functor}$$

These same proofs work for any $f : A \to B$ for which there is a map $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. We get theorems $\forall_g \phi \Vdash_B \star f^\star \phi$ and $f^\star \phi \Vdash_B \exists_g \phi$. These correspond to the theorems about terms substitutable for a given variable.

To get the rule of generalization, classically written as

$$\frac{\phi}{\forall_x \phi}$$

we assume that each of the posets $\mathcal{P}(A)$ has a top element and that $f^\star$ preserves top. The assertion that a predicate of type A is true is then given by $\top \Vdash_A \phi$. Since $f^\star(\top) = \top$, for any $f : A \to B$ we get

$$\frac{\top = f^\star(\top) \Vdash_A \phi}{\top \Vdash_B \forall_f \phi}$$

a variable free form of the rule of generalization.

Several other standard quantification theorems depend on conditions stated as "ψ does not contain any free occurrences of x". In our current setting we do not have variables, so we need to determine how to capture the essential feature of this condition for quantification along a map f. One useful observation is that if ϕ contains no free occurrences of x

then the assignments which take place in an interpretation determine truth values for ϕ independent of the assignment of x. Putting this in the context of quantification along a function we should have a predicate which is not sensitive to application of f. Such predicates are ones of the form $f^\star(\psi)$.

Example. A proof of $\forall_f(f^\star(\phi) \Rightarrow \psi)\Vdash_B \phi \Rightarrow \forall_f(\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\tau\Vdash_B \phi \Rightarrow \forall_f(\psi)}{\tau \wedge \phi\Vdash_B \forall_f(\psi)}\ -\wedge\phi\dashv\phi\Rightarrow -}{f^\star(\tau)\wedge f^\star(\phi) = f^\star(\tau\wedge\phi)\Vdash_A \phi}\ f^\star\dashv\forall_f}{f^\star(\tau)\Vdash_A f^\star(\phi)\Rightarrow\psi}\ -\wedge f^\star(\phi)\dashv f^\star(\phi)\Rightarrow -}{\tau\Vdash_B \forall_f(f^\star(\phi)\Rightarrow\Psi)}\ f^\star\dashv\forall_f$$

If you let τ be $\phi \Rightarrow \forall_f(\psi)$ and read this downward you get a proof of $\phi \Rightarrow \forall_f(\psi)\Vdash_B \forall_f(f^\star(\phi) \Rightarrow \psi)$. If you let τ be $\forall_f(f^\star(\phi) \Rightarrow \psi)$ and read upward (noting that all of the inferences are reversible since all come from adjunctions) you get a proof of $\forall_f(f^\star(\phi) \Rightarrow \psi)\Vdash_B \phi \Rightarrow \forall_f(\psi)$. The key step in the middle is provided by the fact that $f^\star$ preserves conjunction.

Next we will see how to get some of Hájek's axioms on quantifiers [4, p. 111]:

1. ($\forall$1) $(\forall_x \phi(x)) \to \phi(y)$,
2. ($\exists$1) $(\phi(y) \to (\exists_x(\phi(x)))$,
3. ($\forall$2) $\forall_x(\chi \to \phi) \to (\chi \to \forall_x\phi(x))$,
4. ($\exists$2) $(\forall_x(\phi \to \chi)) \to ((\exists_x\phi(x)) \to \chi)$,
5. ($\forall$3) $(\forall_x(\phi \vee \chi)) \to ((\forall_x\phi(x)) \vee \chi)$,

where y is substitutable for x and χ does not contain x freely.

We start by restating the conditions on y and χ:

1. ($\forall$1) $\top\Vdash_T(\forall_!\phi) \to \ulcorner a \urcorner^\star(\phi)$ Here $T \overset{\ulcorner \bullet \urcorner}{\to} A$ and $! : A \to T$ and ϕ is of type A.
2. ($\exists$1) $\ulcorner a \urcorner^\star(\phi)\Vdash_T \exists_!(\phi)$.
3. ($\forall$2) $\forall_f(f^\star(\chi) \to \phi)\Vdash_B \chi \to \forall_f\phi)$ where $f : A \to B$ and χ is of type B.
4. ($\exists$2) $\forall_f(\phi \to f^\star(\chi))\Vdash_B(\exists_f\phi) \to \chi$.
5. ($\forall$3) $\forall_f(\phi \vee f^\star(\chi))\Vdash_B(\forall_f\phi) \vee \chi$.

Example. A proof of $\exists 2 : \forall_f(\phi \to f^\star(\psi))\Vdash_B(\exists_f\phi) \to \psi$.

Here we assume that & is a symmetric monoidal closed structure with exponentiation given by $\to$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall_f(\phi \to f^\star\psi)\Vdash_B \forall_f(\phi \to f^\star\psi)}{f^\star\forall_f(\phi \to f^\star\psi)\Vdash_A \phi \to f^\star\psi}}{\phi\&f^\star\forall_f(\phi \to f^\star\psi)\Vdash_A \phi\&(\phi \to f^\star\psi)} \qquad \dfrac{(\phi \to f^\star\psi)\Vdash_A(\phi \to f^\star\psi)}{\phi\&(\phi \to f^\star\psi)\Vdash_A f^\star\psi}}{\phi\&f^\star\forall_f(\phi \to f^\star(\psi)\Vdash_A f^\star\psi}}{\phi\Vdash_A f^\star(\forall_f(\phi \to f^\star(\psi)) \to \psi) = f^\star\forall_f(\phi \to f^\star\psi) \to f^\star\psi}}{\exists_f\phi\Vdash_B \forall_f(\phi \to f^\star\psi) \to \psi}}{\forall_f(\phi \to f^\star\psi)\&\exists_f\phi = \exists_f\phi\&\forall_f(\phi \to f^\star\psi)\Vdash_B\psi}}{\forall_f(\phi \to f^\star\psi)\Vdash_B(\exists_f\phi) \to \psi}$$

Arguments given earlier give us $\forall$1, $\forall$2, and $\exists$1. The axiom $\forall$3 expresses a distributivity of $\vee$ over $\bigwedge$, an example from [1] shows that linearity of the order on the algebras involved seems to be needed for this axiom. The other direction follows from the calculation given in the next example.

Example. How preservation of operations gives inferences for $\forall$ and $\exists$ If we make the assumption that our functor $f^\star$ preserves the operation $\odot$ (which might be &, $\wedge$, $\vee$, $\Rightarrow$, or $\rightarrow$) then we get the following proofs:

$$\dfrac{\dfrac{\dfrac{\forall_f \phi \Vdash \forall_f \phi}{f^\star \forall_f \phi \Vdash \phi}}{f^\star((\forall_f \phi) \odot v) = f^\star(\forall_f \phi) \odot f^\star(v) \Vdash \phi \odot f^\star(v)}}{(\forall_f \phi) \odot v \Vdash \forall_f(\phi \odot f^\star v)} - \odot f^\star v \text{ functor}$$

and

$$\dfrac{\dfrac{\dfrac{\exists_f \phi \Vdash \exists_f \phi}{\phi \Vdash f^\star \exists_f \phi}}{\phi \odot f^\star v \Vdash f^\star(\exists_f \phi) \odot f^\star v = f^\star(\exists_f \odot v)}}{\exists_F(\phi \odot f^\star v) \Vdash (\exists_f \phi) \odot v}$$

2.2. Change of base: morphisms of predicate logics

To see what a morphism of categorical predicate logics ought to look like let us narrow our consideration to the setting of fuzzy sets with values in complete lattice ordered semigroups, the nicest case for having all the limits and colimits we might need.

If we restrict our attention to a particular set S and look at how variation in the propositional logic affects predicates we again get from a suitable function of lattices $\lambda : L_1 \to L_2$ a trio of functors $\lambda^\uparrow$, λ°, $\lambda^\downarrow$. In the cases of fuzzy sets with values in the lattices these have the following effects:

$$\lambda^\circ : \mathcal{P}_{L_1}(S) \to \mathcal{P}_{L_2}(S) \quad \text{takes } \alpha : S \to L_1 \text{ to } \quad \lambda \circ \alpha : S \to L_1 \to L_2$$

$$\lambda^\uparrow : \mathcal{P}_{L_2}(S) \to \mathcal{P}_{L_1}(S) \quad \text{takes } \beta : S \to L_2 \text{ to } \quad s \mapsto \bigvee \{l \in L_1 \mid \lambda(l) \leq \beta(s)\}$$

$$\lambda^\downarrow : \mathcal{P}_{L_2}(S) \to \mathcal{P}_{L_1}(S) \quad \text{takes } \beta : S \to L_2 \text{ to } \quad s \mapsto \bigwedge \{l \in L_1 \mid \lambda(l) \geq \beta(s)\}$$

Theorem 2. *With these definitions, if λ reflects and preserves order then*

$$\lambda^\uparrow \lambda^\circ(\alpha : A \to L_1)(s) = \alpha(s)$$

and

$$\lambda^\downarrow \lambda^\circ(\alpha : A \to L_1)(s) = \alpha(s)$$

Proof. Suppose $\alpha : S \to L_1$ and $\beta : S \to L_2$. First we calculate

$$\lambda^\uparrow \lambda^\circ(S, \alpha)(s) = \bigvee \{l \in L_1 \mid \lambda(l) \leq \lambda(\alpha(s))\}$$

$$\lambda^\downarrow \lambda^\circ(S, \alpha)(s) = \bigwedge \{l \in L_1 \mid \lambda(l) \geq \lambda(\alpha(s))\}$$

Now if λ reflects and preserves order then

$$\{l \in L_1 \mid \lambda l \leq \lambda(\alpha(s))\} = \{l \in L_1 \mid l \leq \alpha(s)\}$$

and

$$\{l \in L_1 \mid \lambda l \geq \lambda(\alpha(s))\} = \{l \in L_1 \mid l \geq \alpha(s)\}$$

So sups and infs of these pairs of sets will also be equal, giving

$$\begin{aligned} \bigvee \{l \in L_1 \mid \lambda(l) \leq \lambda(\alpha(s))\} &= \bigvee \{l \in L_1 \mid l \leq \alpha(s)\} \\ &= \alpha(s) \\ \bigwedge \{l \in L_1 \mid \lambda(l) \geq \lambda(\alpha(s))\} &= \bigwedge \{l \in L_1 \mid l \geq \alpha(s)\} \\ &= \alpha(s) \quad \square \end{aligned}$$

Theorem 3. *If λ preserves $\bigvee$ then $\lambda^{\uparrow} \dashv \lambda^{\circ}$ and if λ preserves $\bigwedge$ then $\lambda^{\circ} \dashv \lambda^{\downarrow}$.*

Proof. To show the adjointness relations first we calculate

$$\begin{aligned}
\lambda^{\uparrow}\lambda^{\circ}(S,\alpha)(s) &= \bigvee\{l \in L_1 | \lambda(l) \leq \lambda(\alpha(s))\} \\
&\geq \alpha(s) \text{ Since } \alpha(s) \in \{l \in L_1 | \lambda(l) \leq \lambda(\alpha(s))\} \\
\lambda^{\circ}\lambda^{\uparrow}(S,\beta)(s) &= \lambda(\bigvee\{l \in L_1 | \lambda(l) \leq \beta(s)\}) \\
&= \bigvee\{\lambda(l) | \lambda(l) \leq \beta(s)\} \\
&\leq \beta(s)
\end{aligned}$$

This gives $\lambda^{\uparrow} \dashv \lambda^{\circ}$.

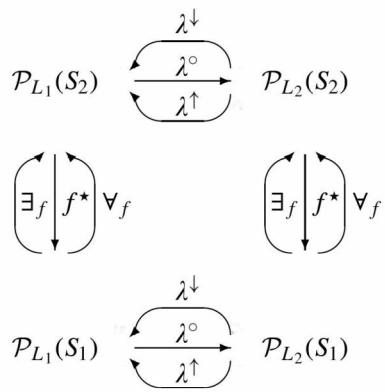
For the other adjointness we calculate

$$\begin{aligned}
\lambda^{\downarrow}\lambda^{\circ}(A,\alpha)(s) &= \bigwedge\{l \in L_1 | \lambda(l) \geq \lambda(\alpha(s))\} \\
&\leq \alpha(s) \\
\lambda^{\circ}\lambda^{\downarrow}(S,\beta)(s) &= \lambda \bigwedge\{l \in L_1 | \lambda(l) \geq \beta(s)\} \\
&= \bigwedge\{\lambda(l) | \lambda(l) \geq \beta(s)\} \\
&\geq \beta(s)
\end{aligned}$$

This gives $\lambda^{\circ} \dashv \lambda^{\downarrow}$. □

These functors play nicely with those giving the predicate logic:

Theorem 4. *In the diagram*



we get:

1. *With no additional hypotheses $f^{\star}\lambda^{\circ} = \lambda^{\circ} f^{\star}$, $f^{\star}\lambda^{\uparrow} = \lambda^{\uparrow} f^{\star}$, and $f^{\star}\lambda^{\downarrow} = \lambda^{\downarrow} f^{\star}$.*
2. *If λ preserves $\bigvee$ then $\exists_f \lambda^{\circ} = \lambda^{\circ} \exists_f$.*
3. If *λ preserves $\bigwedge$ then $\forall_f \lambda^{\circ} = \lambda^{\circ} \forall_f$.*

Proof. These are fairly direct calculations:

$$\begin{aligned}
f^{\star}\lambda^{\circ}(\beta : S_2 \to L_1)(s_1) &= (\lambda \circ \beta)(f(s_1)) \\
\lambda^{\circ} f^{\star}(\beta : S_2 \to L_1)(s_1) &= \lambda(\beta(f(s_1))) \\
&= \lambda^{\circ}\forall_f(\beta : S_2 \to L_1)(s_1)
\end{aligned}$$

$$f^\star\lambda^\downarrow(\gamma : S_2 \to L_2)(s_1) = \lambda^\downarrow(\gamma(f(s_1)))$$
$$= \bigwedge\{l_1 \in L_1 | \lambda(l_1) \geq \gamma(f(s_1))\}$$
$$= \lambda^\downarrow f^\star(\gamma : S_2 \to L_2)(s_1)$$
$$f^\star\lambda^\uparrow(\gamma : S_2 \to L_2)(s_1) = \lambda^\uparrow(\gamma(f(s_1)))$$
$$= \bigvee\{l_1 \in L_1 | \lambda(l_1) \leq \gamma(f(s_1))\}$$
$$= \lambda^\uparrow f^\star(\gamma : S_2 \to L_2)(s_1)$$

When λ preserves $\bigvee$ we get

$$\exists_f \lambda^\circ(\alpha : S_1 \to L_1)(s_2) = \bigvee\{\lambda(\alpha(s_1)) | f(s_1) = s_2\}$$
$$= \lambda \bigvee\{\alpha(s_1) | f(s_1) = s_2\}$$
$$= \lambda^\circ \exists_f(\beta : S_2 \to L_1)(s_1)$$

And when λ preserves $\bigwedge$ we get

$$\forall_f \lambda^\circ(\alpha : S_1 \to L_1)(s_2) = \bigwedge\{\lambda(\alpha(s_1)) | f(s_1) = s_2\}$$
$$= \lambda \bigwedge\{\alpha(s_1) | f(s_1) = s_2\} \quad \square$$

Theorem 5. *In order for λ° to preserve $\wedge$, &, or $\vee$ it suffices for λ to preserve the same operation. Preservation of $\Rightarrow$ calls for λ to reflect order, preserve $\wedge$ and $\bigvee$ and to satisfy a density condition*:

$$\bigvee\{\lambda(l_1), l_1 \in L_1 | \lambda l_1 \leq x\} = \bigvee\{l_2 \in L_2 | l_2 \leq x\}$$

Preservation of $\to$ calls for λ to reflect order, preserve & and $\bigvee$ and satisfy the density condition.

Proof. The situations for $\wedge$, & and $\vee$ are straightforward. The proof for $\Rightarrow$ is given by a calculation:

$$\lambda^\circ(\alpha' \Rightarrow \alpha'')(a) = \lambda(\bigvee\{l \in L_1 | \alpha'(a) \wedge l \leq \alpha''(a)\})$$
$$= \bigvee\{\lambda(l) | \alpha'(a) \wedge l \leq \alpha''(a)\}$$
Since λ preserves $\bigvee$
$$= \bigvee\{\lambda(l) | \lambda(\alpha'(a) \wedge l) \leq \lambda(\alpha''(a))\}$$
Since λ reflects order
$$= \bigvee\{\lambda(l) | \lambda(\alpha'(a)) \wedge \lambda(l) \leq \lambda(\alpha''(a))\}$$
Since λ preserves $\wedge$
$$= \bigvee\{\lambda(l) | \lambda(l) \leq \lambda(\alpha'(a)) \Rightarrow \lambda(\alpha''(a))\}$$
$$= \bigvee\{l_2 \in L_2 | l_2 \leq \lambda(\alpha'(a)) \Rightarrow \lambda(\alpha''(a))\}$$
By the density condition on λ
$$= \bigvee\{l_2 \in L_2 | l_2 \wedge \lambda(\alpha'(a)) \leq \lambda(\alpha''(a))\}$$
$$= \lambda(\alpha'(a)) \Rightarrow \lambda(\alpha''(a))$$

The proof for $\to$ is similar with & replacing $\wedge$. $\square$

The way that the functors λ°, $\lambda^\downarrow$, and $\lambda^\uparrow$ commute with $f^\star$ suggests the following definition of a morphism of categorical logics:

Definition 2. A morphism of categorical logics from $\mathcal{T}_1^{\mathrm{op}} \stackrel{\mathcal{P}_1}{\to}$ **Cat** to $\mathcal{T}_2^{\mathrm{op}} \stackrel{\mathcal{P}_2}{\to}$ **Cat** consists of a functor $G : \mathcal{T}_1 \to \mathcal{T}_2$ and for each object A of $\mathcal{T}$ a functor $g_A : \mathcal{P}_1(A) \to \mathcal{P}_2(G(A))$ such that for any $f : A \to B$ in $\mathcal{T}_1$, the following square commutes:

$$\begin{array}{ccc} \mathcal{P}_1(B) & \stackrel{g_B}{\to} & \mathcal{P}_2(G(B)) \\ f^\star \downarrow & & \downarrow (G(f))^\star \\ \mathcal{P}_1(A) & \stackrel{g_A}{\to} & \mathcal{P}_2(G(A)) \end{array}$$

With this definition we use the identity on **Sets** together with the functors derives from λ to get morphisms back and forth between categorical logics of fuzzy sets with values in different lattices.

Definition 3. A morphism of categorical logics is a morphism of predicate logics if in addition we get the two commutative squares

$$\begin{array}{ccc} \mathcal{P}_1(B) & \stackrel{g_B}{\to} & \mathcal{P}_2(G(B)) \\ \forall_f \uparrow & & \uparrow \forall_{G(f)} \\ \mathcal{P}_1(A) & \stackrel{g_A}{\to} & \mathcal{P}_2(G(A)) \end{array}$$

for the universal fragment and

$$\begin{array}{ccc} \mathcal{P}_1(B) & \stackrel{g_B}{\to} & \mathcal{P}_2(G(B)) \\ \exists_f \uparrow & & \uparrow \exists_{G(f)} \\ \mathcal{P}_1(A) & \stackrel{g_A}{\to} & \mathcal{P}_2(G(A)) \end{array}$$

for the existential fragment.

This is usually too much to ask for. In topos logic, for instance, what we ask for is the preservation of expressions in the geometric logic.

3. Regaining variables: bound and free

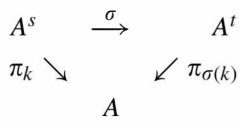
Our presentation so far gives a very large and very general predicate logic. In most settings we are concerned with languages involving only a small number of predicate symbols, a small finite number of variables, and a small finite number of constants. In such a setting we do not need all of **Sets** as underlying category of types. Indeed it is probably sufficient to ask for only a single atomic type (specifying the kind of objects being considered), products of that type with itself up to a small finite length, and maps giving the specified constants.

Suppose that we have a language $\mathfrak{L}$ with n variables $x_1, \ldots, x_n$, k constants $c_1, \ldots c_k$, and m predicates $\phi_1, \ldots, \phi_m$ of arity no more than n. Using this we can produce a category of types. and then build on it a categorical predicate logic freely generated by $\mathfrak{L}$.

Definition 4. The category $\mathcal{M}$ has

- Objects:
  1. the terminal T;
  2. the atomic type A;
  3. for each $s \subseteq \{1, \ldots, n\}$ the product $A^s \simeq A \times \cdots \times A$ with $|s|$ factors. This allows us to keep track of which of the variables appears in the factor.
- Morphisms:
  1. for each object O the unique morphism to the terminal $! : O \to T$;
  2. for each constant c_j a morphism of the same name from T to A;

3. for each s-tuple κ of constants a map $\kappa : T \to A^s$ with $\pi_k \kappa = c_{\kappa(k)}$;
4. for each one to one function $\sigma : t \to s$ where $\emptyset \neq s, t \subseteq \{1, \ldots n\}$ a morphism $\sigma : A^t \to A^s$ with $\pi_{\sigma(k)}\sigma = \pi_k$ in the diagram



   Notice that this gives the identities and the projections;
5. all compositions of morphisms of the kinds already enumerated.

We then define terms as variables or constants. As in the work of Hájek et al. we postpone introduction of function symbols because of the lack of agreement over what fuzzy functions ought to be.

Definition 5. Formulae of type A^s where s is the set of free variables appearing in the formula are defined inductively by:

1. The special constants $\top$ and $\bot$ are predicates of type T.
2. If ϕ is an p-ary predicate and $t_1 \cdots t_p$ are terms then $\phi(t_1, \ldots, t_p)$ is a formula (an atomic formula). Notice that only a finite number of variables are allowed to appear in such an atomic formula and all of them appear free.
3. If Ψ and Ξ are formulae such that no variable appears free in one and bound in the other and x_k is a variable which does not appear bound in Ψ then the following are also formulae:
   (a) $(\Psi \wedge \Xi)$.
   (b) $(\Psi \Rightarrow \Xi)$.
   (c) $(\Psi \& \Xi)$.
   (d) $(\Psi \to \Xi)$.
   (e) $\forall_{x_k} \Psi$. The variable x_k becomes bound.
   (f) $\exists_{x_k} \Psi$. The variable x_k becomes bound.
   (g) If x_l is a variable which does not appear in Ψ then we get a new predicate Ψ_{x_l} in which x_l is free. This is the same predicate as Ψ only it allows explicit consideration of a variable not occurring in Ψ. If s is the set of free variables in Ψ and $\iota : s \to s \cup \{x_l\}$ then $\pi_\iota^\star(\Psi) = \Psi_{x_l}$.

This definition is crafted so that no variable gets reused in the same expression in two different meanings, one bound and one free. All occurrences of a variable are assumed to refer to the same individual. Only a finite number of distinct variables can occur in any formula.

Definition 6. The categories $\mathcal{P}(A^s)$ for a set $s \subseteq \{1, \ldots, n\}$ have as objects the formulae of type A^s and as morphisms all provable inferences $\Phi \Vdash_{A^s} \Psi$ arising from the rules of inference for $\mathcal{P}(A^s)$ being a complete lattice ordered semigroup with each $f^\star$ preserving all limits, colimits, and monoidal structure and with both quantifiers given by adjointness.

The intent here is to give the free categorical predicate logic which allows us to talk about the atomic predicates and constants given. Such a structure can be built for the language of any finite theory by taking the predicates in the theory as atomic and allowing just enough variables to allow us to consider all of the predicates on separate factors.

In a fuzzy setting there are two different possible meanings for non-logical axioms in a theory. They could be given as statements (typically restricted to predicates of type T) assumed to be true (i.e. with $\top \Vdash_T \Psi$) as in classical predicate logic. We will call such a theory a **strong** first order theory. We could also give a set of inferences of the form $\Phi \Vdash_{A^s} \Psi$ involving free variables in s which we could then use as axioms. Such a theory we will call a **weak** first order theory. If the top element of the lattice is always assumed to be the unit for the monoidal structure and the categorical logic is assumed to be truth functional, then weak and strong first order theories will be essentially the same. In any case we

can subsume strong theories in weak ones by replacing a sentence S in the theory which is assumed to be true with an inference $\top \Vdash_T S$.

Using these notions of a first order theory we can define an interpretation of the language of that theory in any categorical predicate logic.

Definition 7. If **T** is a weak first order theory in the language $\mathfrak{L}$ then an interpretation is a morphism of predicate logics from the predicate logic freely generated on $\mathfrak{L}$ to the given categorical predicate logic such that all of the functors g_{A^s} preserve limits, colimits, and the monoidal structure. When the target predicate logic is the truth functional predicate logic of fuzzy sets with values in L we say this is an interpretation in fuzzy sets.

This definition makes the assumption of sufficient completeness so that all of the $f^\star$ functors have both right and left adjoints. As Hájek has noted, for many logics based on infinite, incomplete sublattices of the unit interval this will not be true in general. So one needs *safe* interpretations. Here we will ask for the a morphism of categorical logics from the predicate logic freely generated on $\mathfrak{L}$ which preserve existential and universal quantification when possible. This will mean that some of the expressions in $\mathcal{P}(A^s)$ will not have images under g_{A^s} and as a result we will need to consider the *safe* formulas for the interpretation rather than all of $\mathcal{P}(A^s)$. We will build safe formulas by insisting that each instance of the existential or universal quantification be along a map in the interpretation which is safe in the sense that the relevant limits or colimits exist. Thus both the "free" categorical predicate logic which serves as the domain for the functor giving the interpretation and that interpretation itself need to take into account safety, so safe interpretations are interpretations of a categorical logic which may not have full quantification.

Once we have interpretations we can ask what it takes for an interpretation to be a model of a first order theory.

Definition 8. A model of a strong first order theory T is an interpretation of the language $\mathfrak{L}$ of T in which each axiom gets the value $\top$.

Definition 9. A model of a weak first order theory T is an interpretation of the language $\mathfrak{L}$ of T in which there is a morphism $g_{A^s}(\Phi) \to g_{A^s}(\Psi)$ in $\mathcal{P}(G(A^s))$ for each $\Phi \Vdash_{A^s} \Psi$ in T.

As usual we say the theory T is consistent if there are expressions (or in the case of a weak theory inferences) which are not provable from T.

Completeness will then say that T is consistent if and only if there is a model for T in which $\bot \neq \top$ in the predicates about the terminal.

4. Organizing fuzzy predicate logics categorically

It is common practice in fuzzy set theory to make use of a wide variety of lattices of truth values of a particular kind: there are good reasons to consider all BL-algebras, MV-algebras, Gödel algebras, hoops, or complete lattice ordered semigroups (see [5]). This gives a two-dimensional structure to the categorical setting: the category of algebras used as truth values (recoverable as the predicates of type T) gives one dimension and the category of types gives the other dimension. In general the morphisms of algebras give rise to morphisms of categorical logics with additional preservation properties giving rise to preservation properties for the quantifiers.

We first consider the situation in a Goguen style category similar to that studied by Solovyov in [17] and then look at the categorical fabric I talked about in Linz in 2004.

4.1. Using a variety of lattices as truth values: double fibrations

We can make the whole situation into a category and then give functors to both **Sets** and **CLOSG**, the category of complete lattice ordered semigroups, which are split fibrations:

Definition 10. The category **Set(CLOSG)** has

- as objects triples $(A, L, \alpha : A \to L)$ where A is a set, L is a complete lattice ordered semigroup, and α is a function giving truth values;

- as morphisms from $(A, L_1, \alpha : A \to L_1)$ to $(B, L_2, \beta : B \to L_2)$, pairs of maps $f : A \to B$ in **Sets** and $\lambda : L_1, \to L_2$ in **CLOSG** such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & L_1 \\ f \downarrow & \leq & \downarrow \lambda \\ B & \xrightarrow{\beta} & L_2 \end{array}$$

has $\lambda(\alpha(a)) \leq \beta(f(a))$.

The obvious forgetful functors

$$U_S : \textbf{Set(CLOSG)} \to \textbf{Sets} \text{ and } V_L : \textbf{Set(CLOSG)} \to \textbf{CSLOG}$$

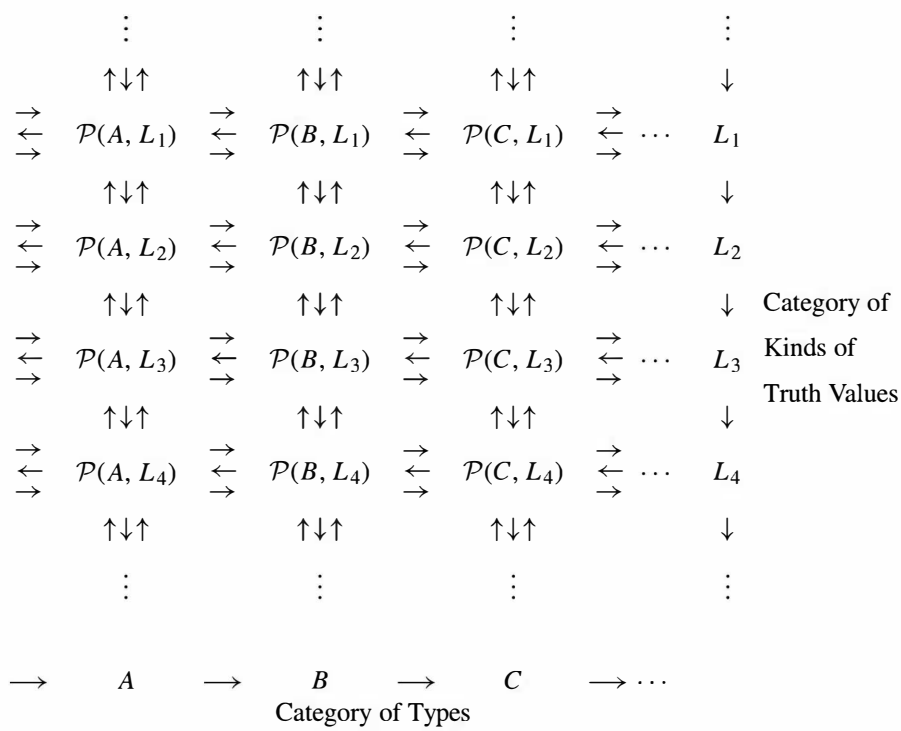
are given by $U_S(A, L_1, \alpha) = A$ and $U_S(f, \lambda) = f$ and $V_L(A, L_1, \alpha) = L_1$ and $V_L(f, \lambda) = \lambda$. The fiber of **Set(CLOSG)** sitting over a specific lattice L and its identity map is the category of fuzzy sets with values in L (the Goguen category). For each set A the fiber gives a family of lattices and functors.

Theorem 6. *The functor U_S is the underlying functor for a split fibration where the choice of Cartesian morphism over $f : A \to B$ with codomain $\beta' : B \to L$ is given by (α', id_L) where $\alpha'(a) = \beta'(f(a))$, giving the functor that we earlier called $f^\star$.*

The functor V_L is the underlying functor for a split fibration where the choice of Cartesian morphism over $\lambda : L_1 \to L_2$ and domain $\alpha : A \to L_1$ is given by $(\mathrm{id}_A, \lambda) : (A, L_1, \alpha) \to (A, L_2, \alpha')$ where $\alpha' = \lambda \circ \alpha : A \to L_2$, giving the functor we earlier called λ°.

4.2. Using a variety of lattices as truth values: categorical fabrics

Theorem 4 tells us how the functors in the picture below play with each other.



Because this picture looks like a woven fabric we make the following definition.

Definition 11. A categorical fabric consists of a category of types $\mathcal{T}$, a category of kinds of targets for truth values $\mathcal{L}$ and for each object L of $\mathcal{L}$ a categorical logic given by categories of predicates of type A given by $\mathcal{P}(A, L)$. These categories of predicates are then connected using functors $f^\star$ and λ° so that the squares

$$\begin{array}{ccc} \mathcal{P}(A, L_1) & \xleftarrow{f^\star} & \mathcal{P}(B, L_1) \\ \lambda^\circ \downarrow & & \downarrow \lambda^\circ \\ \mathcal{P}(A, L_2) & \xleftarrow{f^\star} & \mathcal{P}(B, L_2) \end{array}$$

commute.

If there are adjoints to these functors then the similar squares involving $f^\star$ with $\lambda^\uparrow$ and $\lambda^\downarrow$ commute as do those with $\exists_f$ and $\forall_f$ with λ°. In such a case we say that the fabric is triply woven.

If we take the category of complete lattice ordered semigroups with $\bigvee$ $\bigwedge$ & preserving maps as the category of kinds of truth values and **Sets** as the category of types and use fuzzy subsets as predicate categories we will get a triply woven fabric. That is, however, a much richer environment than most fuzzy set theorists work in and asks rather too much of the maps between kinds of truth values to be fully valuable. We need to consider fabrics with less structure.

In several of the settings Hájek considers the algebras of truth values are only finitely complete, so that the existential and universal quantification functors are only partially defined when the category of types is taken to be **Sets**. Hájek addresses this problem by restricting to *safe* interpretations (those for which all of the limits or colimits called for in the expressions being interpreted do in fact exist). A simpler approach might be to limit the category of types to **Fin**, the category of finite sets. Doing so would make all of the suprema and infima called for in defining the quantifiers finite, and thus guarantee their existence. This would make all of our categorical logics categorical predicate logics. It would not guarantee existence of the functors $\lambda^\downarrow$ or $\lambda^\uparrow$.

Completeness theorems come from the construction of a free category with the kind of structure given and then proof of the existence of morphisms of categorical logics to the standard fabric structures with values in the various kinds of algebras.

Fabrics also provide a possible world semantics for a number of modal logics in which necessity means that a result holds for all descendant fuzzy models with values in the kind of truth values allowed. Possibility means that the result holds for some kind of truth values accessible later in the fabric.

4.3. Kinds of algebras used for truth values

In each of the following the morphisms of the algebras are the functions preserving all of the operations specified in the structure. Categorical predicate logics work best when the algebras are assumed to give lattices closed under both $\bigvee$ and $\bigwedge$ and triply woven fabrics result when these large sups and infs are preserved by the maps of algebras of kind of truth value.

4.3.1. Residuated lattices

Definition 12 (*Hájek [4, p. 47]*). A *residuated lattice* is an algebra

$$(L, \cap, \cup, \star, \Rightarrow, 0, 1)$$

such that

1. $(L, \cap, \cup, 0, 1)$ is a lattice with largest element 1 and smallest element 0.
2. $(L, \star, 1)$ is a commutative semigroup with unit element 1.
3. $x \leq y \Rightarrow z$ if and only if $x \star y \leq z$.

Standard categorical semantics for fuzzy sets with values in a residuated lattice will have each of the categories $\mathcal{P}(A)$ finitely complete and cocomplete and with a symmetric monoidal closed structure arising from the & and the resulting residuation.

Negation in a residuated lattice is given by $\neg x = (x \Rightarrow 0)$.

Höhle [8] removes the condition that the unit for the monoid structure be the top of the lattice and then calls a commutative residuated lattice ordered monoid *integral* if the unit is the top element.

4.3.2. BL algebras

Definition 13 (*Hájek [4, pp. 46–49]*). A *BL algebra* is a residuated lattice such that

1. $x \cap y = x\star(x \Rightarrow y)$ so that $\cap$ is recoverable from $\star$.
2. $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$ prelinearity.

Note that prelinearity follows from divisibility: if $x > y$ then there is a z such that $y = x\star z$. BL algebras give the logic of continuous t-norms.

Definition 14. A *BL_Δ algebra* is a BL algebra such with an additional unary operation Δ such that

1. $\Delta x \cup \neg\Delta x = 1$,
2. $\Delta(x \cup y) \leq \Delta x \cup \Delta y$,
3. $\Delta x \leq x$,
4. $\Delta x \leq \Delta\Delta x$,
5. $(\Delta x)\star(\Delta(x \Rightarrow y)) \leq \Delta y$,
6. $\Delta 1 = 1$.

Here again the categories $\mathcal{P}(A)$ in the standard categorical semantics will be finitely complete and cocomplete and have a symmetric monoidal structure. The additional axioms showing that the join is definable in terms of the monoidal structure needs to be imposed as does semilinearity, though prelinearity follows from truth functionality when the logic over the terminal is linearly ordered.

The Δ operator gives a functor taking a coreflection into a classical logic based subcategory.

4.3.3. MV-algebras

Definition 15 (*Hájek [4, p. 70]*). An *MV algebra* is a BL algebra such that

$$x = (x \Rightarrow 0) \Rightarrow 0$$

An alternate definition is what Hájek calls a Wajsberg algebra:

Definition 16. A *Wajsberg algebra* is an algebra $(A, \Rightarrow, 0)$ such that

1. $(1 \Rightarrow x) = x$,
2. $((x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))) = 1$,
3. $((\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x)) = 1$,
4. $((x \Rightarrow y) \Rightarrow y) = ((y \Rightarrow x) \Rightarrow x)$.

We get an MV algebra by taking

1. $x\star y = \neg(x \Rightarrow \neg y)$,
2. $x \cap y = x\star(x \Rightarrow y)$,
3. $x \cup y = (x \Rightarrow y) \Rightarrow y$.

MV algebras give Łukasiewicz logic.

4.3.4. Π-algebras

The algebras for the product logic are:

Definition 17 (*Hájek [4, p. 89]*). A *Π-algebra* is a BL-algebra satisfying

1. $- - z \leq ((x\star z \Rightarrow y\star z) \Rightarrow (x \Rightarrow y))$,
2. $x \cap -x = 0$.

4.3.5. G-algebras

The Gödel logic gives intuitionistic logic. The relevant algebras are Heyting algebras or G-algebras (Heyting + prelinearity). A G-algebra is a BL-algebra such that

$$x \star x = x$$

4.3.6. Complete lattice ordered semigroups

Goguen developed fuzzy logic in the setting of complete lattice ordered semigroups in [3]. Pavelka gave an extensive further development in the series of papers [14–16]. In [18] categorical constructions for fuzzy logic are carried out over these:

Definition 18. A *complete lattice ordered semigroup* is an algebra $(L, \cup, \cap, \star, 1, 0)$ such that

1. $(L, \cup, \cap, 0, 1)$ is a complete, distributive lattice.
2. $(L, \star, 1)$ is a commutative semigroup (i.e. $\star$ is commutative and associative and has 1 as unit).
3. $a \star \bigvee b_i = \bigvee (a \star b_i)$ so in particular $a \star -$ preserves order.

The completeness combined with the distributivity makes this a residuated lattice using either $\star$ or $\cap$, so we get two distinct implications: $\rightarrow$ adjoint to $\star$ and $\Rightarrow$ adjoint to $\cap$.

The categories $\mathcal{P}(A)$ in the standard semantics with truth values in a complete lattice ordered semigroup are complete, cocomplete, Cartesian closed, symmetric monoidal closed categories. The quantifier functors both exist, so the semantics give a whole categorical predicate logic.

4.3.7. GL-monoids

Definition 19 (*Höhle [9, p. 132]*). A *GL-monoid* is a complete, integral, divisible, residuated, commutative l-monoid (that is, a divisible complete residuated lattice) such that the infinite distributive laws hold:

1. $\alpha \star \bigvee_{i \in I} \beta_i = \bigvee_{i \in I} (\alpha \star \beta_i)$,
2. $\alpha \star \bigwedge_{i \in I} \beta_i = \bigwedge_{i \in I} (\alpha \star \beta_i)$.

The additional requirement here is that $\star$ distribute over $\bigwedge$.

4.3.8. Quantals

While the notion does not originate in the source cited, it is likely to be available to those reading this paper, so look there for further references:

Definition 20 (*Mulvey and Nawaz [13]*). A *quantale* is a lattice having arbitrary joins $\bigvee$ and a (possibly non-commutative) associative product & satisfying the distributive laws:

1. $a \& \bigvee b_i = \bigvee (a \& b_i)$,
2. $(\bigvee b_i) \& a = \bigvee (b_i \& a)$.

A quantale is *unital* if there is a two sided unit for &. Note that it is often the case that the top element is not the unit for &.

If $\top \& a \leq a$ we say a is left-sided. If $a \& \top \leq a$ we say a is right-sided.

The quantale is idempotent if $a \& a = a$ for all a.

A right-sided idempotent quantale is called a *Gelfand* quantale.

Quantale valued fuzzy sets will, in general, have two different implications (adjoint to the two different functors arising from &). The logic there is explored in [19].

The categories $\mathcal{P}(A)$ in a standard unital quantal valued semantics will be complete, finitely cocomplete, and have a monoidal closed structure which is not symmetric.

4.3.9. DeMorgan algebras

In [2] the concentration is on use of min and an idempotent involutive negation. The relevant algebras are:

Definition 21. A *DeMorgan algebra* is a distributive lattice $(A, \vee, \wedge, 0, 1, ')$ such that

1. $(x \vee y)' = x' \wedge y'$,
2. $(x \wedge y)' = x' \vee y'$,
3. $x'' = x$.

Definition 22. A *Kleene algebra* is a DeMorgan algebra satisfying

$$x \wedge x' \leq y \vee y'$$

for all x, y.

A reasonable kind of algebra combines a lattice structure, a monoidal structure, and a DeMorgan structure.

Definition 23. A *fuzzy algebra* is an algebra $(A, \cup, \cap, \star, /, 0, 1)$ satisfying:

1. $(A, \cup, \cap, 0, 1)$ is a complete distributive lattice,
2. $(A, \star, 1)$ is a commutative semigroup,
3. $x \leq y$ if and only if $y' \leq x'$,
4. $x'' = x$,
5. $a \star \bigvee b_i = \bigvee (a \star b_i)$.

These axioms are enough to guarantee that a fuzzy algebra is a residuated lattice. We can recover a t-conorm from a t-norm $\star$ by $a \oplus b = (a' \star b')'$.

The $/$ operation gives a contravariant, idempotent endofunctor on $\mathcal{P}(A)$. The remaining properties for fuzzy algebras show that $\mathcal{P}(A)$ is a complete, cocomplete, Cartesian closed, symmetric monoidal closed category. Taking values in a DeMorgan algebra we may not get the monoidal structure.

4.3.10. Hoops

In [1] Esteva, Godo, Hájek, and Montagna look at a positive fragment of fuzzy logic using the structure of a hoop:

Definition 24. A *hoop* is a structure $(H, \star, \Rightarrow, 1)$ such that $\star$ is a commutative operation with unit 1 and $\Rightarrow$ satisfies:

1. $(x \Rightarrow x) = 1$,
2. $x \star (x \Rightarrow y) = y \star (y \Rightarrow x)$,
3. $(x \star y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$.

Letting $x \leq y$ mean that $(x \Rightarrow y) = 1$ makes this an order which is symmetric monoidal closed as a category (that is, $\star$ respects the order and has a residuation). As a result, the categories $\mathcal{P}(A)$ would be symmetric monoidal closed categories with a terminal object, but generally without an initial object.

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