

## When Does a Category Built on a Lattice with a Monoidal Structure have a Monoidal Structure?

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### Abstract

Finite sets are one of the most fundamental mathematical structures. In the absence of the axiom of choice there are many different inequivalent definitions of finite even in classical logic. When we allow incomplete existence as in fuzzy sets the situation gets even more complicated. This paper gives nine distinct definitions of finite in a fuzzy context together with examples showing how the properties of the underlying lattice of truth values impact the meanings of finite. © 2001 Elsevier Science B.V. All rights reserved.

One of our most fundamental mathematical notions is that of finite set. When I told my colleague Narendra Jaggi, a physicist, the title of the talk this paper is based on his reaction was to say "Trust a mathematician to make the obvious difficult." In some sense that is the problem with finiteness: we are so used to working with finite collections of things in everyday life that the problem seems to be with what *infinite* means rather than with what *finite* means. But everyday life gives us a strong intuition about what finite means, not a rigorous definition we can use to provide a foundation for combinatorial mathematics and arithmetic. Difficulties with finiteness definitions in set theory have been known since the early twentieth century. Tarski's paper [14] gives the classical treatment of several of the variants treated here in the fuzzy case. Rubin's paper [10] gives an easily accessible modern exposition of the equivalence of many notions of finite, updating Tarski's paper. Jech's book on the axiom of choice [6] shows how in the absence of the axiom of choice different definitions are inequivalent.

Combinatorics is the branch of mathematics which deals with finite sets. Its main aim is to talk about how finite sets can be structured to make it possible to count their elements. There are three main principles of counting which form the basis for a starting place on what finiteness should mean:

1. If  $A \cap B = \emptyset$  then  $|A \cup B| = |A| + |B|$ : disjoint cases lead to addition.
2.  $|A \times B| = |A| \times |B|$ : successive choices multiply.
3. If all equivalence classes for an equivalence relation  $N$  on  $A$  have the same number of elements  $m$  then  $|A/N| = |A|/m$ : a systematic over count can be corrected by division.

This tells us to expect that the class of finite sets should be closed under the operations of

1. Disjoint union (coproduct in the categorical setting).
2. Cartesian product (product in the categorical setting).

3. Formation of at least some specified kinds of quotients.

Quotients in fuzzy sets can be particularly bizarre. One reasonable kind of quotient would be an epimorphism with the final structure on its target; that is,  $f: A \rightarrow B$  is epimorphic and  $\beta(b) = \bigvee \{\alpha(a) \mid f(a) = b\}$ .

**Example.** Let  $\mathbf{L}$  be the lattice obtained by taking one chain of length  $n$  (say  $c_{n,1} < c_{n,2} < \dots < c_{n,n}$ ) for each positive  $n \in \mathbb{N}$  and then putting a new top  $\top$  above all the chains and a new bottom  $\perp$  below them. Notice that this lattice satisfies both the ascending and the descending chain condition and that for any  $h \in \mathbf{L}$  with  $h \neq \perp$  the upset from  $h$  given by  $\{k \mid k \geq h\}$  is finite. Let  $A$  be the fuzzy set with carrier  $\{(m, n) \mid 0 < m \leq n \in \mathbb{N}\}$  and  $\alpha(m, n) = c_{n,m}$ . Let  $f: A \rightarrow \mathbb{N}^+$  have  $f(m, n) = m$ . The final structure induced on  $\mathbb{N}^+$  then has all elements sent to  $\top$ . Note that  $f$  is an epimorphism. Note that each of the sets  $\{a \in A \mid \alpha(a) = h\}$  is finite and for  $h > \perp$  the sets  $\{a \in A \mid \alpha(a) \geq h\}$  are also finite. For many of our later definitions the fuzzy set  $(A, \alpha)$  will be finite, but  $(\mathbb{N}^+, \top)$  will never be.

In combinatorics, we also count subsets by considering their characteristic functions, leading us to expect that the power set of a finite set should be finite. In fuzzy sets that is unlikely to be true since it is easy to give examples where the terminal object (the analog of a one point set) has an infinite number of subobjects.

Because the unbalanced subobject weak representer

$$(L, \text{id}_L) \gg \rightarrow (L, \top)$$

is so important, we will check for each definition of finiteness whether or not the generic unbalanced subobject  $(L, \text{id}_L)$  is finite.

The object of this paper is to look at a variety of definitions of finite for fuzzy sets, look at examples to show how far the definitions get from matching our intuition, and establishing which of the combinatorial properties follow for each definition. Assuming the axiom of choice in a two-valued universe, all of the definitions we consider are known to be equivalent. Without the axiom of choice the equivalence breaks down.

Throughout this paper, we will be working in the Goguen category  $\mathbf{Set}(\mathbf{L})$ . A detailed study of its properties can be found in [12]. The objects are pairs  $(A, \alpha)$  where  $A$  is a set and  $\alpha: A \rightarrow L$ . Morphisms  $f: (A, \alpha) \rightarrow (B, \beta)$  are functions  $f: A \rightarrow B$  with  $\beta(f(a)) \geq \alpha(a)$ . A morphism is a monomorphism if the underlying set function is a monomorphism and similarly for epimorphisms. Since we can have different degrees of membership on the same set, it is quite possible for a morphism to be both epic and monic without being an isomorphism. In particular if  $(L, *)$  is a complete lattice-ordered semigroup then  $\mathbf{Set}(\mathbf{L})$  is Cartesian closed (using a product based on  $\wedge$ ), monoidal closed (using a tensor based on  $*$ ), and has weak representation of unbalanced subobjects (fuzzy sets

$$(A, \alpha') \gg \rightarrow (A, \alpha)$$

with the same underlying set, but a potentially smaller degree of membership). The lattice of unbalanced subobjects of  $(A, \alpha)$  is written as  $\mathcal{U}(A, \alpha)$ ; it is represented internally by the powerobject  $P(A, \alpha)$  which consists of the (crisp) set of all unbalanced subobjects of  $(A, \alpha)$ . We assume that negation comes from the residuation of  $*$  rather than from an order reversing involution and a DeMorgan system.

Different definitions of finiteness require different amounts of structure on  $\mathbf{L}$  to give the structures needed in  $\mathbf{Set}(\mathbf{L})$ . For each definition, the minimum structure needed on  $\mathbf{L}$  will be given.

## 1. Natural numbers and cardinal finiteness

While the fuzzy real line and the fuzzy unit interval have long histories in the literature of fuzzy sets, there is little written about fuzzy natural numbers. Indeed, the usual practice is to make the natural numbers crisp.

Following the usual practice in mathematical logic, I will use the natural numbers as those which are thought of as counting the number of elements in classical finite sets, thus starting at 0 (the number of elements in the empty set) rather than at 1:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

If  $\mathbf{L}$  is a partially ordered set with top element  $\top$  then we will think of  $(\mathbb{N}, \top)$  as the natural numbers in

**Set(L)**. Using this we can say what cardinal finiteness means:

**Definition 1.** A finite cardinal  $[n]$  is  $\{x \mid x \in \mathbb{N}, x < n\}$ . Thus,

$$\begin{aligned} [0] &= \emptyset, \\ [1] &= \{0\}, \\ &\vdots \\ [n] &= \{0, 1, 2, \dots, n-1\}. \end{aligned}$$

So  $[n]$  has exactly  $n$  elements.

As the first step towards defining finite in a fuzzy setting, we can take the strong form.

**Definition 2.** A fuzzy set  $(S, \sigma)$  is cardinal finite if it is isomorphic to  $[n]$  for some  $n \in \mathbb{N}$ .

Here isomorphism asks for a map in **Set(L)** which has an inverse in **Set(L)**. Since **Set(L)** is not a balanced category this is not the same as asking for a function which is both monic and epic, since those functions are allowed to increase the degree of membership in fuzzy sets, while isomorphisms must preserve the degree of membership exactly.

**Proposition 1.** *Cardinal finite fuzzy sets are crisp sets with finite underlying set. The class of cardinal finite fuzzy sets is closed under coproducts, products, and quotients, but not under unbalanced subobjects.*

**Proof.** Because isomorphisms must preserve degree of membership exactly, the cardinal finite fuzzy sets are just crisp finite sets. If  $\mathbf{L}$  is a lattice, then the class of crisp finite sets is closed under coproducts and products, but it is not closed under formation of fuzzy subsets (unbalanced subobjects in **Set(L)**) or fuzzy power object formation. Epimorphic images of crisp sets are again crisp, so quotients of cardinal finite fuzzy sets will again be cardinal finite.  $\square$

The cardinality of a cardinal finite set is a crisp natural number, the number of elements in its carrier.

Cardinal finiteness captures none of the fuzziness of the category **Set(L)**, and thus is not a very good candidate for what finite should mean in a fuzzy context. An improvement comes from requiring a mor-

phism which is both epic and monic rather than an isomorphism:

**Definition 3.** A fuzzy set  $(S, \sigma)$  is weakly cardinal finite if there is a monic and epic map from  $(S, \sigma)$  to  $[n]$  for some  $n \in \mathbb{N}$ .

What this does is close the class of cardinal finite fuzzy sets under formation of unbalanced subobjects. A fuzzy set  $(S, \sigma)$  will then be weakly cardinal finite precisely when  $S$  is a (cardinal) finite set.

**Proposition 2.** *The class of weakly cardinal finite fuzzy sets is closed under coproducts, products, unbalanced subobjects, and quotients. In general, if  $\mathbf{L}$  is infinite, it will not be closed under unbalanced power object formation, since  $\mathbf{L}$  is recovered as the power object of the terminal object  $(\{*\}, \top)$  and the terminal object is clearly cardinal finite.*

The cardinality of a weakly cardinal finite set would be given by a function from  $\mathbf{L}$  to  $[[S]]$  taking each  $h \in \mathbf{L}$  to the number of elements of  $S$  with membership at least  $h$ . Any non-increasing function from  $\mathbf{L}$  to a finite cardinal  $[n]$  can appear as cardinality of a weakly cardinal finite set, so such functions can be identified as fuzzy natural numbers for this class of finite objects.

## 2. Dedekind finiteness

Dedekind [4] gave a definition of *simply infinite* sets as ones for which a one-to-one function  $\phi : S \rightarrow S$  could be found which missed a point  $p$  in  $S$ . He then constructed a sequence of distinct elements of  $S$  by letting  $a_0 = p$  and  $a_{n+1} = \phi(a_n)$ . A set  $S$  then becomes finite if there are no one-to-one functions (monomorphisms) from  $S$  to a proper subset of  $S$ .

There are (at least) two ways to state this positively:

**Definition 4.** A fuzzy set  $(S, \sigma)$  is strongly Dedekind finite if every monomorphism  $m : (S, \sigma) \rightarrow (S, \sigma)$  is in fact an isomorphism.

**Definition 5.** A fuzzy set  $(S, \sigma)$  is Dedekind finite if every monomorphism  $m : (S, \sigma) \rightarrow (S, \sigma)$  is also an epimorphism.

Notice that strong Dedekind finiteness implies Dedekind finiteness.

**Example.** A Dedekind finite set can have infinite support: In  $\mathbf{Set}([0, 1])$  consider the fuzzy set  $(A, \alpha)$  with  $A = \{(1/2^n) \mid n \in \mathbb{N}\}$  and  $\alpha(a) = a$ . A monomorphism  $m$  from  $(A, \alpha)$  to itself must map each  $1/2^n$  to an element  $1/2^m$  with  $m \leq n$  in order to be a map in  $\mathbf{Set}([0, 1])$ . Since there is no larger value to go to, 1 must be fixed. Thus  $\frac{1}{2}$  must also be fixed since the only places it can go are 1 and  $\frac{1}{2}$  and 1 has already been used. An induction argument will show that each  $1/2^n$  must be fixed by  $m$  since if we define them in order there is only one choice at each step. Thus, the only monomorphism from  $(A, \alpha)$  to itself in  $\mathbf{Set}([0, 1])$  is the identity. Thus,  $(A, \alpha)$  is strongly Dedekind finite. This tells us that fuzziness can impose restrictions on the monic endomorphisms not present at the  $\mathbf{Set}$  level.

**Proposition 3.** *If any level set  $S_h = \{s \in S \mid \sigma(s) = h\}$  is infinite then  $(S, \sigma)$  is not Dedekind finite.*

**Proof.** Let  $s_1, s_2, \dots$  be an infinite sequence of elements of  $S$  with  $\sigma(s_i) = h$ . Define a map  $f : (S, \sigma) \rightarrow (S, \sigma)$  as fixing all elements not in the sequence and by taking  $f(s_i) = s_{i+1}$ . This is monic, but not epic. Thus,  $(S, \sigma)$  is neither strongly Dedekind finite nor Dedekind finite.  $\square$

**Proposition 4.** *If  $L$  satisfies an ascending chain condition every ascending chain  $h_1 \leq h_2 \leq \dots$  has a maximum element and every level set  $S_h$  of  $(S, \sigma)$  is finite, then  $(S, \sigma)$  is Dedekind finite.*

**Proof.** Suppose that  $(S, \sigma)$  is not Dedekind finite. Then there is a monomorphism  $m : (S, \sigma) \rightarrow (S, \sigma)$  which is not epic. Let  $s$  be such that no  $t \in S$  has  $m(t) = s$  and look at the sequence of distinct values  $s, m(s), m(m(s)), \dots, m^n(s), \dots$ . The sequence  $\sigma(s), \sigma(m(s)), \dots, \sigma(m^n(s)), \dots$  is an ascending sequence in  $L$  and thus must have a maximum element  $h$ . Thus, for some  $k$  all of the  $m^n(s)$  for  $n > k$  have  $\sigma(m^n(s)) = h$ . This tells us that  $S_h$  is infinite.  $\square$

**Example.** The example given earlier for the pathology of quotients was built on a lattice which satisfies the ascending chain condition and had each  $A_h$  finite, so  $(A, \alpha)$  is Dedekind finite. The quotient map

$f : (A, \alpha) \rightarrow (\mathbb{N}^+, \top)$  gives a quotient which is not Dedekind finite.

**Example.** Without the ascending chain condition finite level sets will not suffice: For  $L = [0, 1]$  the fuzzy set  $(L, \text{id}_L)$ , the generic unbalanced subobject, has every level set having a unique element. However, the function

$$s : [0, 1] \rightarrow [0, 1], \\ h \mapsto \sqrt{h}$$

is an endomorphism increasing degree of membership which is both monic and epic, but is not an isomorphism so  $(L, \text{id}_L)$  is not strongly Dedekind finite. The function

$$f : [0, 1] \rightarrow [0, 1], \\ x \mapsto \frac{1}{2}x + \frac{1}{2}$$

is a monic endomorphism increasing degree of membership which is not epic, so  $(L, \text{id}_L)$  is not Dedekind finite either.

Closure properties for Dedekind finite objects can be bad as the following examples illustrate:

**Example.** Suppose  $L$  is the chain of elements of  $[0, 1]$  either of the form  $1/2^n$  for  $n \in \mathbb{N}$  or of the form  $\frac{1}{2} + 1/2^{n+1}$  or 0. Then  $L$  satisfies the ascending chain condition, so if we take the fuzzy set  $(A, \alpha)$  with one distinct element of each degree of membership we will get a Dedekind finite fuzzy set. We can define a lattice-ordered semigroup structure by defining

$$a * b = \begin{cases} b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ \frac{1}{2} & \text{if both } \frac{1}{2} \leq a < 1 \text{ and } \frac{1}{2} \leq b < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using this structure the fuzzy set  $(A, \alpha) \otimes (A, \alpha)$  is not Dedekind finite since it has an infinite number of elements at the  $\frac{1}{2}$  level.

**Example.** A similar problem with products arises in fuzzy sets with values in the (non-distributive) lattice with a top, then an infinite number of mutually incomparable levels, then a middle value (say  $\frac{1}{2}$ ) and then the bottom. This also satisfies an ascending chain

condition, so if we take a single distinct element at each level of membership we will get a Dedekind finite object  $(B, \beta)$  for which  $(B, \beta) \times (B, \beta)$  has an infinite number of elements at level of membership  $\frac{1}{2}$  and thus is not Dedekind finite.

Dedekind finiteness was studied in a topos setting in [11]. In that paper the notion of equifibered quotient is used. An equifibered quotient  $f : (A, \alpha) \rightarrow (B, \beta)$  is one for which every monic map  $m : (B, \beta) \rightarrow (B, \beta)$  lifts to a monic  $\hat{m} : (A, \alpha) \rightarrow (A, \alpha)$ . Any equifibered quotient of a Dedekind finite object is again Dedekind finite.

### 3. Kuratowski finiteness

Kuratowski observed that finiteness could be determined by looking at the semilattice structure of the powerset. Here, we need for  $\mathbf{L}$  to be at least an  $\vee$ -semilattice. In which case the unbalanced subobjects of a fuzzy set  $(S, \sigma)$  will also form an  $\vee$ -semilattice  $\mathcal{U}(S, \sigma)$ , so we can ask about the  $\vee$ -semilattice generated by the singletons.

**Definition 6.** The smallest  $\vee$ -semilattice of  $\mathcal{U}(S, \sigma)$  containing the singletons is called  $K(S, \sigma)$ . If  $K(S, \sigma)$  has  $(S, \sigma)$  as a member, then we call  $(S, \sigma)$  Kuratowski finite.

This notion of finiteness is shown to be equivalent to several other formulations in [9] for objects in a topos.

Kuratowski finite fuzzy sets are built using pairwise max from singletons. Singletons in  $(S, \sigma)$  are fuzzy subsets of the form  $(S, \chi_s)$  where

$$\chi_s(x) = \begin{cases} \sigma(s) & \text{if } x = s, \\ \perp & \text{otherwise.} \end{cases}$$

The smallest  $\vee$ -semilattice of  $\mathcal{U}(S, \sigma)$  containing the singletons,  $K(S, \sigma)$ , consists of fuzzy subsets  $(S, \sigma')$  where  $\sigma'(x) = \perp$  for all but a finite number of elements of  $S$  and where it is not  $\perp$ ,  $\sigma'(x) = \sigma(x)$ . Asking that  $(S, \sigma)$  be in  $K(S, \sigma)$  is precisely asking for the support of  $(S, \sigma)$  to be finite. Kuratowski and weak cardinal finiteness almost coincide in fuzzy sets: while a weak cardinal finite fuzzy set must have a finite underlying set, a Kuratowski finite fuzzy set could have an infinite

number of elements with degree of membership  $\perp$ . This characterization tells us that the class of Kuratowski finite objects will be closed under products, sums, tensor product, unbalanced subobjects, and quotients.

In topos theory, the decidable Kuratowski finite objects are the ones usually taken as most desirable. Here, decidable means that the diagonal  $X \rightarrow X \times X$  is complemented. See [7] and [1] for details. Since for most notions of negation the only fuzzy sets which are decidable are crisp, this notion is not useful in a fuzzy context.

Bornological spaces have been studied in several works by Hogbe-Nlend [5]. A bornology on a set  $S$  has the same closure properties as the generalize set of bounded subsets of  $S$ .

**Definition 7.** A bornology on a set  $S$  is a collection  $B$  of subsets such that

1.  $\bigcup B = S$ .
2. If  $S' \in B$  and  $S'' \in B$  then  $S' \cup S'' \in B$ .
3. If  $S' \in B$  and  $S'' \subset S'$  then  $S'' \in B$ .

Any set has a trivial bornology, the powerset itself. In an infinite set, the set of finite subsets will be a bornology. In  $\mathbb{R}$ , we could use the set of bounded subsets.

**Definition 8.** A set is bornologically finite if it admits only one bornology.

In **Sets**, bornological finiteness and Kuratowski finiteness are essentially the same since any one point subset must be bounded if the union of all the bounded subsets of  $S$  is to be all of  $S$ . Having the bornology closed under pairwise unions then forces the Kuratowski finite subsets to be bounded. This is the only bornology if it contains  $S$  as a bounded set.

In **Set(L)** this is no longer the case as the following example shows:

**Example.** Let  $\mathbf{L}$  be the unit interval  $[0, 1]$  with the usual order as lattice structure. Then there is a non-trivial bornology on  $(\{\star\}, 1)$  given by the collection of fuzzy subsets  $(\{\star\}, h)$  with  $h < 1$ . This is closed under pairwise  $\vee$  and has  $\bigvee_{h < 1} (\{\star\}, h) = (\{\star\}, 1)$  because 1 is the supremum of the numbers less than 1. A similar construction of a non-trivial bornology can

be made whenever  $L$  has an element which is the supremum of elements strictly smaller than itself. Thus, in the category of fuzzy sets on such a lattice the terminal object is not Bornologically finite.

#### 4. Order-related forms of finiteness

Existence of certain kinds of orderings also characterizes finite sets:

**Definition 9.** A set  $S$  is Stäckel finite if it can be doubly well ordered; i.e. there is an order relation  $\leq$  on  $S$  such that if  $S'$  is a non-empty subset of  $S$ , then  $S'$  has both a greatest and a least element with respect to the order  $\leq$ .

This definition was used by Brook [3] in topoi.

This definition is difficult to use in fuzzy sets because there are multiple possible meanings for “non-empty” and “element”.

**Definition 10.** The degree of non-emptiness of a fuzzy set  $(A, \alpha)$  is

$$\pi(\alpha) = \bigvee_{a \in A} \alpha(a).$$

To define well ordering in a fuzzy context we will need to internalize “if  $A'$  is non-empty then it has a least element” by saying that the extent to which  $A'$  is non-empty implies the extent to which it has a least element. Thus if there is an element with degree of membership  $h > \perp$  then there must be an element with degree of membership at least  $h$  which is smaller than all other elements of degree of membership at least  $h$ . The elements of degree  $\perp$  are ignored.

Having a double well ordering then asks that for every fuzzy subset  $(A, \alpha')$  and element  $a \in A$  with  $\alpha'(a) > \perp$ , the set  $\{t \mid \alpha'(t) \geq \alpha'(a)\}$  has both a smallest and a largest element. This happens precisely when each of the sets  $\{a \in A \mid \alpha(a) \geq h\}$  for  $h > \perp$  is finite.

**Proposition 5.** A fuzzy set  $(A, \alpha)$  is Stäckel finite if and only if each of the sets (the  $h$ -cuts)  $A^h = \{a \in A \mid \alpha(a) \geq h\}$  for  $h > \perp$  is finite.

**Corollary 6.** The generic unbalanced subobject  $(L, \text{id}_L)$  is Stäckel finite if and only if  $L$  has each set of the form  $\{h' \geq h\}$  finite for  $h > \perp$ .

**Corollary 7.** The Stäckel finite fuzzy sets are closed under disjoint sum, tensor product, and product.

**Proof.** This follows from inequalities on  $h$ -cuts:

$$(A + B)^h = A^h + B^h,$$

$$(A \otimes B)^h \subseteq A^h \times B^h,$$

$$(A \times B)^h \subseteq A^h \times B^h.$$

The latter two follow since if  $h \leq h_1 * h_2$  then  $h \leq h_1$  and  $h \leq h_2$ . This tells us that the  $h$ -cuts of the sum and product will be finite if the  $h$ -cuts of the original fuzzy sets were.  $\square$

**Example.** The example given earlier for the pathology of quotients had each  $A^h$  finite for  $h > \perp$ , so  $(A, \alpha)$  is Stäckel finite. The quotient map  $f : (A, \alpha) \rightarrow (\mathbb{N}^+, \top)$  gives a quotient which is not Stäckel finite.

**Example.** Suppose  $L$  is the chain in  $[0, 1]$  consisting of  $1, 0$ , and all the points of the form  $1/2^n$ . This satisfies an ascending chain condition. Suppose that  $(S, \sigma)$  has one element with degree of membership  $1/2^n$  for each  $n$  and an infinite number of elements with degree of membership  $0$ . Then  $(S, \sigma)$  is Stäckel finite, but it is neither Dedekind finite nor Kuratowski finite.

**Example.** A Dedekind finite fuzzy set need not be Stäckel finite. Let  $L$  be the ordinal number  $\omega \cdot \omega$  with the reverse order. Then  $L$  satisfies the ascending chain condition. Thus,  $(L, \text{id}_L)$  is Dedekind finite. It has, however,  $L^h$  infinite for  $h > \omega$ , so it is not Stäckel finite.

#### 5. Tarski's definitions

**Definition 11.** A set  $S$  is Tarski finite if every non-empty family  $\mathcal{F}$  of subsets of  $S$  has an irreducible element; that is, an element  $A \in \mathcal{F}$  such that if  $A' \in \mathcal{F}$  and  $A' \subseteq A$  then  $A' = A$ .

**Example.** For fuzzy sets on  $[0, 1]$  even the terminal object  $(\star, 1)$  is not Tarski finite: Let  $A_k$  be the fuzzy

set on one element with membership level  $1/2^k$ . Then the chain

$$(\star, 1) \supset A_1 \supset A_2 \supset A_3 \supset \dots$$

has no irreducible element.

Tarski shows that for sets this is equivalent to what Jech calls T-finite:

**Definition 12** (Jech [6]).  $S$  is T-finite if every non-empty monotone subset  $A \subseteq \mathcal{P}(S)$  has a  $\subseteq$ -maximal element.

**Example.** For fuzzy sets on  $[0, 1]$  even the terminal object  $(\star, 1)$  is not T-finite: Let  $A_k$  be the fuzzy set on one element with membership level  $1 - 1/2^k$  for  $k \geq 1$ . Then the chain

$$A_1 \subset A_2 \subset A_3 \subset \dots$$

has no maximal element.

It is easy to construct examples to demonstrate that T-finiteness and Tarski finiteness are distinct in fuzzy sets: to make the terminal object T-finite it is necessary and sufficient for  $\mathbf{L}$  to satisfy an ascending chain condition; to make it Tarski finite what is needed is a descending chain condition.

Since these notions of finiteness depend so heavily on the finiteness conditions satisfied by  $\mathbf{L}$  they tend to confound the properties of  $\mathbf{L}$  with the properties of objects in  $\mathbf{Set}(\mathbf{L})$ .

If  $\mathbf{L}$  is finite, then T-finite fuzzy sets and Tarski finite fuzzy sets are those with every level set  $S_h$  finite, coinciding in this situation with the weakly cardinal finite fuzzy sets.

If  $\mathbf{L}$  satisfies an ascending chain condition, then T-finiteness and Dedekind finiteness agree.

If  $\mathbf{L}$  satisfies a descending chain condition, then a fuzzy set  $(S, \sigma)$  will be Tarski finite if and only if each  $S_h$  is finite. Such fuzzy sets need not satisfy any of the other finiteness conditions.

## 6. Ultrafiniteness

In order to talk about filters, we will ask that  $(\mathbf{L}, *)$  be a complete lattice ordered semigroup. The follow-

ing definition is from [13]. Recall that  $P(A, \alpha)$  is the crisp set of unbalanced subobjects of  $(A, \alpha)$ .

**Definition 13.** A filter on  $(A, \alpha)$  is a function  $\phi: P(A, \alpha) \rightarrow \mathbf{L}$  such that

1.  $\phi$  preserves order: if  $\alpha' \geq \alpha''$  then  $\phi(\alpha') \geq \phi(\alpha'')$ .
2.  $\phi$  respects  $*$ :  $\phi(\alpha' * \alpha'') \geq \phi(\alpha') * \phi(\alpha'')$ .
3. A fuzzy set can only be in a filter to its degree of non-emptiness:  $\phi(\alpha') \leq n(\alpha')$ . This makes our filters proper.

For filter-based definitions of finite we also need to know what principle filters are and what an ultrafilter is. Note that the non-emptiness condition cannot be strengthened if we want the principle filter to be a filter:

**Definition 14.** The principle filter  $\phi_a$  has  $\phi_a(\alpha') = \alpha'(a)$ .

In the absence of an order reversing involution giving a complement, the notion of ultrafilter is given by a maximality condition on covering pairs. In [2, p. 60] this is given as an equivalent to the definition as a maximal filter and in [8] it is used to define ultrafilters on lattices.

**Definition 15.** A filter  $\phi$  is an ultrafilter if it satisfies the additional condition  $\phi(\alpha' \vee \alpha'') = \phi(\alpha') \vee \phi(\alpha'')$ .

Notice that we always have  $\phi(\alpha' \vee \alpha'') \geq \phi(\alpha') \vee \phi(\alpha'')$  by the order-preserving property. Thus, the ultrafilter condition can be thought of either as requiring that the  $\phi(\alpha')$  and  $\phi(\alpha'')$  be large or requiring that  $\phi(\alpha' \vee \alpha'')$  be small.

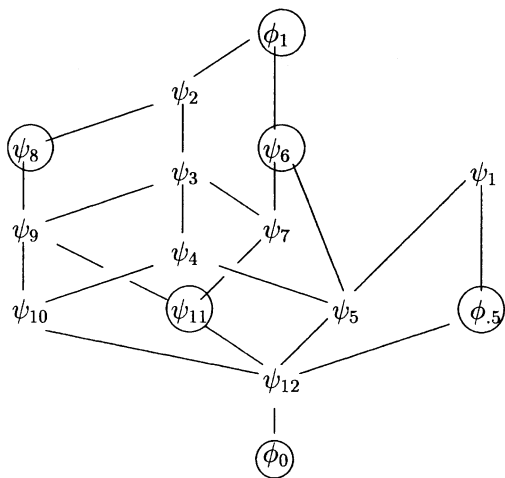
If  $\mathbf{L}$  has a weak form of complements so that for any  $h \geq k \in \mathbf{L}$  there is a  $k' \in \mathbf{L}$  with  $h = k \vee k'$ , then this condition is the same as asking that for any unbalanced subobject of  $(A, \alpha)$  either the subobject or its weak complement is in the filter. Most lattices used for fuzzy sets do not have this kind of weak complement, so the description in terms of covering pairs is preferable to a description using complements.

With this definition every principle filter is in fact an ultrafilter. The following example, however, shows that principle filters need not be maximal (and thus, that ultrafilters using this definition need not be maximal either!)

**Example.** Let  $L$  be the chain  $0 < 0.5 < 1$  using  $* = \wedge$  and consider filters on the fuzzy set  $(L, \text{id}_L)$ . The powerobject of  $(L, \text{id}_L)$  has six elements, so we can give all filters explicitly using a table, saving a little space by noting that 0 always goes to 0:

	$\begin{pmatrix} 1 & 0.5 \\ 1 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 \\ 0 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix}$
$\psi_1$	1	0.5	0.5	0	0	0
$\phi_1$	1	0.5	0	1	0.5	0
$\psi_2$	1	0.5	0	0.5	0.5	0
$\psi_3$	1	0.5	0	0.5	0	0
$\psi_4$	1	0.5	0	0	0	0
$\psi_5$	1	0	0	0	0	0
$\psi_6$	1	0	0	1	0	0
$\psi_7$	1	0	0	0.5	0	0
$\phi_{0.5}$	0.5	0.5	0.5	0	0	0
$\psi_8$	0.5	0.5	0	0.5	0.5	0
$\psi_9$	0.5	0.5	0	0.5	0	0
$\psi_{10}$	0.5	0.5	0	0	0	0
$\psi_{11}$	0.5	0	0	0.5	0	0
$\psi_{12}$	0.5	0	0	0	0	0
$\phi_0$	0	0	0	0	0	0

These form a partially ordered set with Hasse diagram (here the ultrafilters are circled).



Notice that this example shows that ultrafilters using this definition need not be maximal; maximal filters need not be ultrafilters, and neither maximal filters nor ultrafilters need to be principle.

Part of the difficulty in this example came from the existence of elements with partial membership. To tie ultrafilters in our sense in with usual ultrafilters we should consider sets which are, if not crisp, then as crisp as they can be:

**Definition 16.** A fuzzy set  $(A, \alpha)$  is  $h$ -crisp if  $\alpha$  assumes only the values  $h$  and  $\perp$ . Each  $h$ -crisp fuzzy subset corresponds to a subset  $A' \subseteq A$  consisting of the elements sent to  $h$ .

**Proposition 8.** If  $\psi$  is a fuzzy ultrafilter on the fuzzy set  $(A, h)$ , then the  $h$ -crisp fuzzy subsets of  $A$  which have  $\psi(\alpha) = h$  induce an ultrafilter on  $A$  in **Sets**.

**Proof.** The fuzzy filter conditions give us a family of subsets closed under superset and intersection, hence a filter. The ultrafilter condition makes it a prime filter in **Sets**. Assuming the axiom of choice in **Sets** this is sufficient to get an ultrafilter.  $\square$

In **Sets** any infinite set  $S$  has an ultrafilter refining the filterbase of the subsets of  $S$  which have finite complements. This ultrafilter is not principle. In finite sets, however, the only ultrafilters are the principle ones, leading to the following definition:

**Definition 17.** A fuzzy set  $(A, \alpha)$  is ultrafinitary if every ultrafilter on  $(A, \alpha)$  is principle.

Compactness can be defined in terms of the convergence of ultrafilters as in Bourbaki. A topological structure on a fuzzy set  $(A, \alpha)$  will give rise to a neighborhood filter on each  $a \in A$  which is contained in the principle filter on  $a$ . A filter converges to  $a$  if it contains the neighborhood filter of  $a$ . A space is compact if every ultrafilter converges. In a fuzzy setting these ideas need fully fuzzy topology as in [13].

**Proposition 9.** A fuzzy set is ultrafinitary if and only if every topology on it is compact.

**Proof.** Saying that every ultrafilter is principle is precisely the same as saying that the discrete topology on  $(A, \alpha)$  is compact, so the only if part is clear. Since neighborhood filters are always contained in principle filters, if we know every ultrafilter is principle, then we will know that every ultrafilter converges.  $\square$

In topos theory, this definition was considered by Volger [15] and is rejected by Johnstone [7] because the subobject representer in a topos is always ultrafinitary.



## 7. Summary of relationships between the definitions

**Theorem 10.** *For fuzzy sets with truth values in a lattice  $\mathbf{L}$*

1. *Cardinal finite implies Weak cardinal finite.*
2. *Weak cardinal finite implies Kuratowski finite and both forms of Dedekind finite.*
3. *Kuratowski finite implies Stäckel finite.*
4. *Bornological finite implies Kuratowski finite with the reverse implication holding only if no element of  $\mathbf{L}$  is the supremum of strictly smaller elements.*

*In general, none of these implications are reversible and there is no relationship between Dedekind finite and Stäckel finite. Tarski finiteness and  $T$ -finiteness are acceptable only if  $\mathbf{L}$  satisfies appropriate chain conditions.*

## 8. Summary of the closure properties

Because the sum in  $\mathbf{Set}(\mathbf{L})$  is computed as a disjoint union, nearly all of the structures needed in defining finiteness are well behaved with respect to sums. In particular, the level sets of a sum are given by the disjoint union of the level sets of the pieces and the  $h$ -cut of  $(A, \alpha) + (B, \beta)$  is the product of the  $h$ -cut of  $(A, \alpha)$  with the  $h$ -cut of  $(B, \beta)$  so finiteness conditions on level sets or  $h$ -cuts will be preserved. Similarly, chain conditions on the subobjects of a sum will follow from chain conditions on the subobjects of the summands.

Finiteness notion	Closed under			Structures finite			
	$\times$	$\otimes$	$+$	Unbalanced Quotients	$(L, id_L)$	Terminal subobject	
Cardinal	Y	Y	Y	N	Y	N	Y
Weak cardinal	Y	Y	Y	Y	Y	N	Y
Dedekind	N	N	Y		N	N	N
Kuratowski	Y	Y	Y	Y	Y	N	Y
Stäckel	Y	Y	Y	Y	N	N	Y

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