On the Convergence of the Summation Formulas Constructed by Using a Symbolic Operator Approach

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Abstract

This paper deals with the convergence of the summation of power series of the form $S_a^b(f;x) = \sum_{a\le k\le b} f(k)x^k$, where $0 \le a < b \le \infty$, and $\{f(k)\}$ is a given sequence of numbers with $k \in [a,b)$ or $f(t)$ a differentiable function defined on $[a,b)$. Here the summation is found by using the symbolic operator approach shown in [4] . We will give a different type of the remainder of the summation formulas. The convergence of the corresponding power series will be determined consequently. Some examples such as the generalized Euler's transformation series will also be given. In addition, we will compare the convergence of the given series transforms.

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1 Introduction

In [4], we present a symbolic summation operator with its various expansions, and construct several summation formulas with estimable remainders for $S_a^b(f;x) = \sum_{a\le k\le b} f(k)x^k$, with the aid of some classical interpolation series due to Newton, Gauss and Everett, respectively, where $0 \le a < b \le \infty$, and $\{f(k)\}$ is a given sequence of numbers with $k \in [a,b)$ or $f(t)$ a differentiable function defined on $[a,b)$. In order to discuss the convergence of the summation formulas, we will give a new type of remainders.

We now start from the following notations. It is known that the symbolic operations Δ (difference), E (displacement) and D (derivative) play important roles in the Calculus of Finite Differences as well as in certain topics of Computational Methods. For various classical results, see e.g., Jordan [6], Milne-Thomson [8], etc. Certainly, the theoretical basis of the symbolic methods could be found within the theory of formal power series, in as much as all the symbolic expressions treated are expressible as power series in Δ , E or D, and all the operations employed are just the same as those applied to formal power series. For some easily accessible references on formal series, we recommend Bourbaki [1], Comtet [2] and Wilf [10].

Recall that the operators Δ , E and D may be defined via the following relations:

$$\Delta f(t) = f(t+1) - f(t), \quad Ef(t) = f(t+1), \quad Df(t) = \frac{d}{dt}f(t).$$

Using the number 1 as an identity operator, viz. $1f(t) = f(t)$, one can observe that these operators satisfy the formal relations

$$E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1+\Delta).$$

Powers of these operators are defined in the usual way. In particular, one may define for any real number x , viz., $E^x f(t) = f(t+x)$.

Note that $E^k f(0) = \left[E^k f(t)\right]_{t=0} = f(k)$, so that any power series of the form $\sum_{k=0}^{\infty} f(k)x^k$ could be written symbolically as

$$\sum_{k\geq 0} f(k)x^k = \sum_{k\geq 0} x^k E^k f(0) = \sum_{k\geq 0} (xE)^k f(0) = (1-xE)^{-1} f(0).$$

This shows that the symbolic operator $(1-xE)^{-1}$ with parameter x can be applied to $f(t)$ (at $t=0$) to yield a power series or a generating function (GF) for $\{f(k)\}$. We shall need several definitions as follows.

Definition 1.1 *The expression $f(t) \in C^m_{[a,b)}$ $(m \geq 1)$ means that $f(t)$ is a real function continuous together with its mth derivative on $[a,b)$.*

Definition 1.2 *$\alpha_k(x)$ is called an Eulerian fraction and may be expressed in the form (cf. Comtet [2])*

$$\alpha_k(x) = \frac{A(x)}{(1-x)^{k+1}}, \quad (x \neq 1),$$

where $A_k(x)$ is the kth degree Eulerian polynomial having the expression

$$A_k(x) = \sum_{j=1}^{k} A(k,j)x^j, \quad A_0(x) \equiv 1$$

with $A(k,j)$ being known as Eulerian numbers, expressible as

$$A(k,j) = \sum_{i=0}^{j} (-1)^i \binom{k+1}{i} (j-i)^k, \quad (1 \leq j \leq k).$$

Definition 1.3 *δ is Sheppard central difference operator defined by the relation $\delta f(t) = f\left(t+\frac{1}{2}\right) - f\left(t-\frac{1}{2}\right)$, so that (cf. Jordan [6])*

$$\delta = \Delta E^{-1/2} = \Delta / E^{1/2}, \quad \delta^{2k} = \Delta^{2k} E^{-k}.$$

Definition 1.4 *A sequence $\{x_n\}$ will be called a null sequence if for any given positive (rational) number ϵ, there exists an integer n_0 such that for every $n > n_0$ $|x_n| < \epsilon$.*

For null sequences we quote the following result from [7] (see Theorem 4 in §43).

Lemma 1.5 *Let $\{x_0, x_1, \cdots\}$ be a null sequence and suppose the coefficients $a_{n,\ell}$ of the system $A = \{a_{i,j} : 0 \le j \le i; i = 0, 1, 2, \ldots\}$ satisfy the two conditions:*

(i) Every column contains a null sequence, i.e., for fixed $p \ge 0$, $a_{n,p} \to 0$ when $n \to \infty$.

(ii) There exists a constant K such that the sum of the absolute values of the terms in any row, i.e., for every n, the sum $|a_{n,0}| + |a_{n,1}| + \ldots + |a_{n,n}| < K$.

Then the sequence formed by the numbers $x'_n = a_{n,0}x_0 + a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n$ is also a null sequence.

Obviously, Lemma 1.5 is a consequence of the Toeplitz Theorem (*cf.* P.74 in [7]).

Definition 1.6 *For any real or complex series $\sum_{k=0}^{\infty} a_k$, the so-called Cauchy root is defined by $r = \overline{\lim}_{k\to\infty}|a_k|^{1/k}$. Clearly, $\sum_{k=0}^{\infty} a_k$ converges absolutely whenever $r < 1$.*

In [4], we have shown in §3 that $(1 - xE)^{-1}$ could be expanded into series in various ways to derive various symbolic operational formulas as well as summation formulas for $\sum_{k\ge 0} f(k)x^k$. For the completeness of the paper, we now cite the result as follows.

Proposition 1.7 *Let $\{f(k)\}$ be a given sequence of (real or complex) numbers, and let $g(t)$ be infinitely differentiable at $t = 0$. Then we have formally*

$$\sum_{k=0}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) \tag{1.1}$$

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(\delta^{2k} f(1) - x\delta^{2k} f(0)\right) \tag{1.2}$$

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(x^{-1}\delta^{2k} f(0) - \delta^{2k} f(-1)\right) \tag{1.3}$$

$$\sum_{k=0}^{\infty} g(k)x^k = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} D^k g(0) \tag{1.4}$$

where we always assume that $x \ne 0$ and $x \ne 1$.

Remark 1.1 We may write (*cf.* [5] and [9]) the Eulerian fractions $\alpha_k(x)$ as

$$\alpha_k(x) = \sum_{j=0}^{k} j! S(k,j) \frac{x^j}{(1-x)^{j+1}}, \tag{1.5}$$

where $S(k,j)$ are Stirling numbers of the second kind. Substituting Eq. (1.5) into Eq. (1.4) and noting $j! \sum_{k=j}^{\infty} S(k,j) \frac{D^k}{k!} = (e^D - 1)^j = \Delta^j$ (*cf.* [1] and [4]) yields

$$\begin{aligned}
\sum_{k=0}^{\infty} g(k) x^k &= \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} D^k g(0) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{j!}{k!} S(k,j) \frac{x^j}{(1-x)^{j+1}} D^k g(0) \\
&= \sum_{j=0}^{\infty} \frac{x^j}{(1-x)^{j+1}} \left(j! \sum_{k=j}^{\infty} S(k,j) \frac{D^k}{k!} \right) g(0) \\
&= \sum_{j=0}^{\infty} \frac{x^j}{(1-x)^{j+1}} \Delta^j g(0),
\end{aligned}$$

which is the series expansion (1.1). Hence, in this paper, we only discuss the convergence of (1.1)-(1.3).

We shall give a new type of remainder in Section 2 for each of the summation formulas shown in the series transforms in Proposition 1.7. In Section 3, we shall discuss the convergence of the summation formulas by using the established remainders. Some examples such as the generalized Euler's transformation series will also be given. In addition, we will compare the convergence of the given series transforms.

2 Summation Formulas with Remainders

In this section we will establish three summation formulas with remainders.

Theorem 2.1 *Let $\{f(k)\}$ be a given sequence of numbers (real or complex). Then we have formally*

$$\sum_{k=0}^{\infty} f(k)x^k = \sum_{k=0}^{n-1} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) + \frac{x^n}{(1-x)^n} \sum_{\ell=0}^{\infty} x^\ell \Delta^n f(\ell). \tag{2.1}$$

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{n-1} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(\delta^{2k} f(1) - x\delta^{2k} f(0)\right) + \left(\frac{x}{(1-x)^2}\right)^n \sum_{\ell=1}^{\infty} x^\ell \delta^{2n} f(\ell) \tag{2.2}$$

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{n-1} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(x^{-1}\delta^{2k} f(0) - \delta^{2k} f(-1)\right) + \left(\frac{x}{(1-x)^2}\right)^n \sum_{\ell=0}^{\infty} x^\ell \delta^{2n} f(\ell). \tag{2.3}$$

where we always assume that $x \neq 0$.

Proof. From Eq. (1.1) we obtain

$$\begin{aligned}
(1-xE)^{-1} = (1-x-x\Delta)^{-1} &= (1-x)^{-1}\left(1 - \frac{x}{1-x}\Delta\right)^{-1} \\
&= (1-x)^{-1}\left\{\sum_{\ell=0}^{n-1}\left(\frac{x}{1-x}\right)^\ell \Delta^\ell + \frac{\left(\frac{x}{1-x}\Delta\right)^n}{1-\left(\frac{x}{1-x}\Delta\right)}\right\} \\
&= \sum_{\ell=0}^{n-1} \frac{x^\ell}{(1-x)^{\ell+1}}\Delta^\ell + \left(\frac{x}{1-x}\right)^n \frac{\Delta^n}{1-xE} \\
&= \sum_{\ell=0}^{n-1} \frac{x^\ell}{(1-x)^{\ell+1}}\Delta^\ell + \left(\frac{x}{1-x}\right)^n \sum_{\ell=0}^{\infty} x^\ell E^\ell \Delta^n.
\end{aligned}$$

Since $E^\ell \Delta^n f(0) = \Delta^n E^\ell f(0) = \Delta^n f(\ell)$, applying operator $(1-xE)^{-1}$ and the rightmost operator shown in the above equalities to $f(t)|_{t=0}$ yields Eq. (2.1).

Similarly, we can derive formula (2.2) formally as follows. From Eq. (1.2), we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} x^k f(k) \\
= &\sum_{k=1}^{n-1} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(\delta^{2k} f(1) - x\delta^{2k} f(0)\right) \\
&+ \sum_{k=n}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \Delta^{2k} E^{-k} (E-x) f(0).
\end{aligned}$$

Applying $\Delta = E - 1$ to the last series yields

$$\begin{aligned}
&\sum_{k=n}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \Delta^{2k} E^{-k} (E-x) f(0) \\
= &\sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+n+1} \left(\frac{\Delta^2}{E}\right)^{k+n} (E-x) f(0) \\
= &\left(\frac{x}{(1-x)^2}\right)^{n+1} \left(\frac{\Delta^2}{E}\right)^{n} \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k} \left(\frac{\Delta^2}{E}\right)^{k} (E-x) f(0) \\
= &\left(\frac{x}{(1-x)^2}\right)^{n+1} \left(\frac{\Delta^2}{E}\right)^{n} \frac{E-x}{1 - \frac{x}{(1-x)^2}\frac{\Delta^2}{E}} f(0) \\
= &\left(\frac{x}{(1-x)^2}\right)^{n} \left(\frac{\Delta^2}{E}\right)^{n} \frac{x(E-x)}{(1-x)^2 - x\frac{\Delta^2}{E}} f(0) \\
= &\left(\frac{x}{(1-x)^2}\right)^{n} \left(\frac{\Delta^2}{E}\right)^{n} \frac{Ex(E-x)}{(1-x)^2 E - x(E-1)^2} f(0) \\
= &\left(\frac{x}{(1-x)^2}\right)^{n} \left(\frac{\Delta^2}{E}\right)^{n} \frac{Ex}{1-xE} f(0) \\
= &\left(\frac{x}{(1-x)^2}\right)^{n} \sum_{k=1}^{\infty} x^k \delta^{2n} f(k),
\end{aligned}$$

which implies Eq. (2.2). Eq. (2.3) can be derived by using a similar argument.

Remark 2.1 In Eq. (2.1), if we assume $x = -1$, then we have the following Euler's series transform

$$\sum_{k=0}^{\infty} x^k f(k) = \sum_{k=0}^{n-1} \frac{(-1)^k}{2^{k+1}} \Delta^k f(0) + \frac{(-1)^n}{2^n} \sum_{\ell=0}^{\infty} (-1)^\ell \Delta^n f(\ell). \qquad (2.4)$$

Hence, we may call (2.1) the generalized Euler's series transform, which can be used to accelerate the series convergence.

We now use Theorem 2.1 to discuss the convergence of the transformation series (2.1)-(2.3).

Theorem 2.2 *Let $\{f(k)\}$ be a given sequence of numbers (in $\mathbb{R}$ or $\mathbb{C}$), and let $\theta = \overline{\lim_{k\to\infty}} |f(k)|^{1/k}$. Then for any given x with $x \neq 0$ we have the convergent expressions (2.1), (2.2) and (2.3), provided that $\theta < 1/|x|$.*

Proof. Suppose that the condition $\theta < 1/|x|$ ($x \neq 0$) is fulfilled, so that $\theta|x| < 1$. Hence the convergence of the series on the left-hand side of (2.1)-(2.3) is obvious in accordance with the Cauchy's root test.

To prove the convergence of the right-hand side of (2.1), it is sufficient to show that $\sum_{\ell=0}^{\infty} x^\ell \Delta^n f(\ell)$ is absolutely convergent. Choose $\rho > \theta$ such that

$$\theta|x| < \rho|x| < 1.$$

Thus, for large k we have $f(k)|^{1/k} < \rho$, i.e., $|f(k)| < \rho^k$. Consequently we have, for large ℓ

$$\begin{aligned} |\Delta^n f(\ell)|^{1/\ell} &\leq \left(\sum_{j=0}^{n} \binom{n}{j} |f(\ell+j)| \right)^{1/\ell} \leq \left(\sum_{j=0}^{n} \binom{n}{j} \rho^{\ell+j} \right)^{1/\ell} \\ &= \rho(1+\rho)^{n/\ell} \to \rho \end{aligned}$$

as $\ell \to \infty$. Thus

$$\overline{\lim_{\ell\to\infty}} \left| x^\ell \Delta^n f(\ell) \right|^{1/\ell} \leq \rho|x| < 1,$$

so that the series on the right-hand side of (2.1) is also convergent absolutely.

The absolute convergence of the right-hand side series in (2.2) and (2.3) can be proved similarly. The only difference is to estimate $|\delta^{2n} f(\ell)|^{1/\ell}$ which can be done as follows under the same condition $\theta < 1/|x|$.

$$\begin{aligned} & \left|\delta^{2n} f(\ell)\right|^{1/\ell} = \left|\Delta^{2n} f(\ell-n)\right|^{1/\ell} \\ \leq & \left(\sum_{j=0}^{2n} \binom{2n}{j} |f(\ell-n+j)|\right)^{1/\ell} \leq \left(\sum_{j=0}^{2n} \binom{2n}{j} \rho^{\ell-n+j}\right)^{1/\ell} \\ = & \rho^{1-n/\ell}(1+\rho)^{n/\ell} \to \rho \end{aligned}$$

as $\ell \to \infty$. It follows that

$$\overline{\lim_{\ell\to\infty}} \left|x^\ell \delta^{2n} f(\ell)\right|^{1/\ell} \leq \rho|x| < 1,$$

which implies the absolute convergence of the right-hand side series of both (2.2) and (2.3).

The argument in Theorem 2.2 applies to negative values of x with $x \geq -1$. Thus for $x = -1$ we have the following corollary.

Corollary 2.3 *Let* $\overline{\lim_{k\to\infty}} |f(k)|^{1/k} < 1$. *Then we have the convergent series*

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k f(k) &= \sum_{k=0}^{n-1} (-1)^k \frac{\Delta^k f(0)}{2^{k+1}} \\ &\quad + \frac{(-1)^n}{2^n} \sum_{\ell=0}^{\infty} (-1)^\ell \Delta^n f(\ell) \qquad (2.5) \\ \sum_{k=0}^{\infty} (-1)^k f(k) &= \sum_{k=0}^{n-1} (-1)^{k+1} \frac{\delta^{2k} f(1) + \delta^{2k} f(0)}{4^{k+1}} \\ &\quad + \frac{(-1)^n}{4^n} \sum_{\ell=1}^{\infty} (-1)^\ell \delta^{2n} f(\ell) \qquad (2.6) \\ \sum_{k=0}^{\infty} (-1)^k f(k) &= \sum_{k=0}^{n-1} (-1)^k \frac{\delta^{2k} f(0) + \delta^{2k} f(-1)}{4^{k+1}} \\ &\quad + \frac{(-1)^n}{4^n} \sum_{\ell=0}^{\infty} (-1)^\ell \delta^{2n} f(\ell) \qquad (2.7) \end{aligned}$$

The condition shown in Theorem 2.2 can be replaced by the following weaker condition.

Theorem 2.4 *Let $\{f(k)\}$ be a given sequence of numbers (real or complex) such that $\sum_{k=0}^{\infty} f(k)x^k$ is convergent. Then we have convergent expressions (2.1), (2.2), and (2.3) for every $x < 0$.*

Proof. We write the remainder of expression (2.1) as follows.

$$\begin{aligned} & R_n := \frac{x^n}{(1-x)^n} \sum_{\ell=0}^{\infty} x^\ell \Delta^n f(\ell) \\ = \; & \frac{x^n}{(1-x)^n} \sum_{\ell=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} x^\ell \binom{n}{j} f(j+\ell) \\ = \; & \frac{(-x)^n}{(1-x)^n} \sum_{j=0}^{n} \binom{n}{j} \sum_{\ell=0}^{\infty} (-1)^j x^\ell \binom{n}{j} f(j+\ell) \\ = \; & \frac{(-x)^n}{(1-x)^n} \sum_{j=0}^{n} (-x)^{-j} \binom{n}{j} \sum_{\ell=j}^{\infty} x^\ell f(\ell) = \frac{(-x)^n}{(1-x)^n} \sum_{j=0}^{n} (-x)^{-j} \binom{n}{j} x_j, \end{aligned}$$

where $x_j = \sum_{\ell=j}^{\infty} x^\ell f(\ell)$ $(0 \le j \le n)$. Since $\sum_{\ell=0}^{\infty} x^\ell f(\ell)$ converges, x_j is the term of a null sequence (see Definition 1.4). To apply Lemma 1.5 here, we consider the coefficients

$$a_{n,j} := \frac{(-x)^n}{(1-x)^n} (-x)^{-j} \binom{n}{j}.$$

Hence, if j is fixed, for every $x \in [-1, 0)$ we have $a_{n,j} \to 0$ as $n \to \infty$ because of

$$|a_{n,j}| = \frac{|x|^{n-j}}{(1-x)^n} \binom{n}{j} < \frac{n^j}{(1-x)^n}$$

and $1/(1-x) < 1$, and for every $x < -1$ we also have $a_{n,j} \to 0$ when $n \to \infty$, seeing that it is

$$|a_{n,j}| = \frac{|x|^{n-j}}{(1-x)^n} \binom{n}{j} < \left(\frac{|x|}{1-x} \right)^n n^j$$

and $|x/(1-x)| < 1$. In addition, for every n and for every $x < 0$ we have

$$\sum_{j=0}^{n} |a_{n,j}| = \frac{1}{(1-x)^n} \sum_{j=0}^{n} (-x)^{n-j} \binom{n}{j} = 1.$$

Therefore, from Lemma 1.5, we find that R_n is also the term of a null sequence, so the series on the right-hand side of (2.5) converges for every $x < 0$. Using the same argument, we can show the convergence of the series on the right-hand side of Eqs. (2.6) and (2.7) for every $x < 0$.

Remark 2.2 If $x < 0$, the convergence rate of the right-hand side series of either (2.2) or (2.3) is faster than the convergence rate of the right-hand series of (2.1) because the rate of the former two series are $O((x/(1-x)^2)^n)$ while the rate of the latter is $O((x/(1-x))^n)$, where $|x/(1-x)^2| < |x/(1-x)| < 1$.

It is easy to see that the convergence of the series shown in (2.1)-(2.3) depends on both the property of f and the range of x (i.e., the convergence interval). From Theorems 2.2 and 2.4, we find that to ensure the convergence, more stringent requirements on f allow for weaker demands on the range of x, and the reverse is also true. However, the expressions of (2.1)-(2.3) show that the largest possible convergence interval for x is $x < 1/2$. To prove it, we need an alternate approach that will be shown in the next section.

3 An alternate approach for the convergence

We will give other convergence conditions for series (2.1)-(2.3). The series can be derived with the aid of the symbolic computation or more formally with the use of some identities, in which the largest possible convergence intervals for x can be shown.

Theorem 3.1 *Let $\{f(k)\}$ be a given sequence of numbers (real or complex). Then we have formally Eq. (2.1) for every $x < 1/2$, Eq. (2.2) for every $x > 3+2\sqrt{2}$ or $x < 3-2\sqrt{2}$, and Eq. (2.3) for every $x > 3+2\sqrt{2}$ or $x < 3-2\sqrt{2}$ and $x \neq 0$.*

Proof. From Eq. (1.1) we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} x^k f(k) &= \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) \\ &= \sum_{k=0}^{n-1} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) + \sum_{k=n}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0). \end{aligned}$$

Noting $\Delta = E - 1$, we write the last summation as

$$\begin{aligned}
& \sum_{k=n}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) \\
= & \sum_{k=0}^{\infty} \frac{x^{n+k}}{(1-x)^{n+k+1}} \Delta^n ((E-1)^k f(0)) \\
= & \sum_{k=0}^{\infty} \frac{x^{n+k}}{(1-x)^{n+k+1}} \Delta^n \left(\sum_{\ell=0}^{k} \binom{k}{\ell} E^\ell (-1)^{k-\ell} f(0) \right) \\
= & \frac{x^n}{(1-x)^{n+1}} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{x^k}{(1-x)^k} (-1)^{k-\ell} \binom{k}{\ell} \Delta^n f(\ell) \\
= & \frac{x^n}{(1-x)^{n+1}} \sum_{\ell=0}^{\infty} (-1)^\ell \Delta^n f(\ell) \left(\sum_{k=\ell}^{\infty} \frac{(-x)^k}{(1-x)^k} \binom{k}{\ell} \right) \\
= & \frac{x^n}{(1-x)^{n+1}} \sum_{\ell=0}^{\infty} (-1)^\ell \Delta^n f(\ell) \left(\sum_{k=0}^{\infty} \left(\frac{-x}{(1-x)} \right)^{k+\ell} \binom{k+\ell}{\ell} \right) \qquad (3.1)
\end{aligned}$$

By using the well-known summation formula (see e.g., [3], (1.3))

$$\sum_{\ell=0}^{\infty} z^k \binom{k+\ell}{\ell} = \frac{1}{(1-z)^{k+1}}, \quad |z| < 1,$$

we change the last series in (3.1) to be

$$\frac{x^n}{(1-x)^{n+1}} \sum_{\ell=0}^{\infty} \Delta^n f(\ell) \left(\frac{x}{(1-x)} \right)^\ell \frac{1}{\left(1 - \frac{-x}{1-x}\right)^{\ell+1}},$$

for $x < 1/2$, which is equivalent to the remainder shown in (2.1).

We now derive formula (2.2). From Eq. (1.2), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} f(k) x^k = & \sum_{k=1}^{n-1} \left(\frac{x}{(1-x)^2} \right)^{k+1} \left(\delta^{2k} f(1) - x \delta^{2k} f(0) \right) \\
& + \sum_{k=n}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{k+1} \Delta^{2k} E^{-k} (E-x) f(0). \qquad (3.2)
\end{aligned}$$

Applying $\Delta = E - 1$ to the last series yields

$$\begin{aligned}
& \sum_{k=n}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{k+1} \Delta^{2k} E^{-k} (E-x) f(0) \\
= & \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{n+k+1} \Delta^{2n} E^{-n-k} (E-x)(E-1)^{2k} f(0) \\
= & \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{n+k+1} \Delta^{2n} E^{-n} (E-x) \left(\sum_{u=0}^{2k} \binom{2k}{u} E^{u-k} (-1)^u \right) f(0) \\
= & \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{n+k+1} \Delta^{2n} E^{-n} (E-x) \left(\sum_{\ell=-k}^{k} \binom{2k}{\ell+k} E^{\ell} (-1)^{\ell+k} \right) f(0).
\end{aligned}$$

We split the last summation of the rightmost equality and obtain

$$\sum_{k=n}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{k+1} \Delta^{2k} E^{-k} (E-x) f(0) = \phi_1(f) + \phi_2(f),$$

where

$$\begin{aligned}
& \phi_1(f) \\
:= & \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{n+k+1} \Delta^{2n} E^{-n} (E-x) \left(\sum_{\ell=-k}^{0} \binom{2k}{\ell+k} E^{\ell} (-1)^{\ell+k} \right) f(0) \\
= & \left(\frac{x}{(1-x)^2} \right)^{n+1} \sum_{\ell=-\infty}^{0} (-1)^{\ell} \delta^{2n} E^{\ell} (E-x) f(0) \sum_{k=-\ell}^{\infty} \binom{2k}{k+\ell} (-1)^k \left(\frac{x}{(1-x)^2} \right)^k \\
= & \left(\frac{x}{(1-x)^2} \right)^{n+1} \sum_{\ell=-\infty}^{0} (-1)^{\ell} \delta^{2n} E^{\ell} (E-x) f(0) \sum_{k=0}^{\infty} \binom{2k-2\ell}{k} \left(\frac{-x}{(1-x)^2} \right)^{k-\ell}
\end{aligned}$$

and

$$
\begin{aligned}
& \phi_2(f) \\
:= & \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{n+k+1} \Delta^{2n} E^{-n}(E-x) \left(\sum_{\ell=1}^{k} \binom{2k}{\ell+k} E^{\ell}(-1)^{\ell+k}\right) f(0) \\
= & \left(\frac{x}{(1-x)^2}\right)^{n+1} \sum_{\ell=1}^{\infty} (-1)^{\ell} \delta^{2n} E^{\ell}(E-x) f(0) \sum_{k=\ell}^{\infty} \binom{2k}{k-\ell} (-1)^{k} \left(\frac{x}{(1-x)^2}\right)^{k} \\
= & \sum_{\ell=1}^{\infty} (-1)^{\ell} \delta^{2n} (f(\ell+1) - x f(\ell)) \sum_{k=0}^{\infty} \binom{2k+2\ell}{k} \left(\frac{-x}{(1-x)^2}\right)^{k+\ell}.
\end{aligned}
$$

We now apply the Polya and Szegö identity (see e.g., [3] (1.120))

$$\sum_{k=0}^{\infty} \binom{a+bk}{k} z^k = \frac{(1-x)^{a+1}}{1-(1-b)x}$$

to $\phi_1(f)$ and $\phi_2(f)$ with $z = -x/(1-x)^b$, $b = 2$, and $a = 2\ell$ and -2ℓ, respectively, where $|z| < |(b-1)^{b-1}/b^b| = 1/4$ or equivalently $x > 3 - 2\sqrt{2}$ or $x < 3 - 2\sqrt{2}$. Here, we need $x \neq -1$. However, this limitation will be omitted after we combine the resulting expressions of $\phi_1(f)$ and $\phi_2(f)$ later.

Substituting the Polya and Szegö identity into the last expressions of $\phi_1(f)$ and $\phi_2(f)$ yields respectively

$$
\begin{aligned}
& \phi_1(f) \\
= & \left(\frac{x}{(1-x)^2}\right)^{n+1} \sum_{\ell=-\infty}^{0} (-1)^{\ell} \delta^{2n} (f(\ell+1) - x f(\ell)) \left(\frac{-x}{(1-x)^2}\right)^{-\ell} \frac{(1-x)^{-2\ell+1}}{1+x} \\
= & \left(\frac{x}{(1-x)^2}\right)^{n+1} \sum_{\ell=0}^{\infty} (-1)^{\ell} \delta^{2n} (f(-\ell+1) - x f(-\ell)) \frac{(1-x)(-x)^{\ell}}{1+x}
\end{aligned}
$$

and

$$
\begin{aligned}
& \phi_2(f) \\
= & \left(\frac{x}{(1-x)^2}\right)^{n+1} \sum_{\ell=1}^{\infty} (-1)^{\ell} \delta^{2n} (f(\ell+1) - x f(\ell)) \left(\frac{-x}{(1-x)^2}\right)^{\ell} \frac{(1-x)^{2\ell+1}}{1+x} \\
= & \left(\frac{x}{(1-x)^2}\right)^{n+1} \sum_{\ell=1}^{\infty} (-1)^{\ell} \delta^{2n} (f(\ell+1) - x f(\ell)) \frac{(1-x)(-x)^{\ell}}{1+x}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\sum_{k=n}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}\Delta^{2k}E^{-k}(E-x)f(0)=\phi_1(f)+\phi_2(f)\\
=\;&\frac{x^{n+1}}{(1+x)(1-x)^{2n+1}}\left\{\sum_{\ell=0}^{\infty}x^{\ell}\delta^{2n}(f(-\ell+1)-xf(-\ell))\right.\\
&\left.+\sum_{\ell=1}^{\infty}x^{\ell}\delta^{2n}(f(\ell+1)-xf(\ell))\right\}\\
=\;&\frac{x}{1-x^2}\left(\frac{x}{(1-x)^2}\right)^{n}\left\{\sum_{\ell=1}^{\infty}x^{\ell}\delta^{2n}(f(-\ell+1)-xf(-\ell))\right.\\
&\left.+\sum_{\ell=0}^{\infty}x^{\ell}\delta^{2n}(f(\ell+1)-xf(\ell))\right\}\\
=\;&\frac{x}{1-x^2}\left(\frac{x}{(1-x)^2}\right)^{n}\left\{\sum_{\ell=0}^{\infty}x^{\ell+1}\delta^{2n}f(-\ell)-\sum_{\ell=1}^{\infty}x^{\ell+1}\delta^{2n}f(-\ell)\right.\\
&\left.+\sum_{\ell=1}^{\infty}x^{\ell-1}\delta^{2n}f(\ell)-\sum_{\ell=0}^{\infty}x^{\ell+1}\delta^{2n}f(\ell)\right\}\\
=\;&\left(\frac{x}{(1-x)^2}\right)^{n}\sum_{\ell=1}^{\infty}x^{\ell}\delta^{2n}f(\ell). \qquad (3.3)
\end{aligned}$$

The rightmost equality shows that the limitation $x \neq -1$ is no longer needed. Hence, we obtain (2.2), which holds for all x that satisfies either $x > 3+2\sqrt{2}$ or $x < 3-2\sqrt{2}$. Similarly, we can derive (2.3) and it completes the proof of Theorem 3.1.

To give a compressed form of the remainders of the transform series shown as (2.1)-(2.3), we need the following lemma (see also in [4]).

Lemma 3.2 *(Mean Value Theorem) Let $\sum_{n=0}^{\infty}a_nx^n$ with $a_n \geq 0$ be a convergent series for $x \in (0,1)$. Suppose that $\phi(t)$ is a bounded continuous function on $(-\infty,\infty)$, and $\{t_n\}$ is a sequence of real numbers. Then there is a number $\xi \in (-\infty,\infty)$ such that*

$$\sum_{n=0}^{\infty}a_n\phi(t_n)x^n=\phi(\xi)\sum_{n=0}^{\infty}a_nx^n.$$

Theorem 3.3 *Let $f(t)$ be a bounded continuous function on $(-\infty, \infty)$. Then for $x < 1/2$*

$$\sum_{k=0}^{\infty} f(k)x^k = \sum_{k=0}^{n-1} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) + \frac{x^n}{(1-x)^{n+1}} \Delta^n f(\xi), \tag{3.4}$$

where $\xi \in (-\infty, \infty)$.

For $x > 3 + 2\sqrt{2}$ or $x < 3 - 2\sqrt{2}$

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{n-1} \left(\frac{x}{(1-x)^2} \right)^{k+1} \left(\delta^{2k} f(1) - x\delta^{2k} f(0) \right) + \frac{x^{n+1}}{(1-x)^{2n+1}} \delta^{2n} f(\xi), \tag{3.5}$$

where $\xi \in (-\infty, \infty)$.

Finally, for $x > 3 + 2\sqrt{2}$ or $x < 3 - 2\sqrt{2}$ and $x \neq 0$, we have

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{n-1} \left(\frac{x}{(1-x)^2} \right)^{k+1} \left(x^{-1}\delta^{2k} f(0) - \delta^{2k} f(-1) \right) + \frac{x^n}{(1-x)^{2n+1}} \delta^{2n} f(\xi), \tag{3.6}$$

where $\xi \in (-\infty, \infty)$.

Proof. Clearly, (3.4)-(3.6) are merely consequences of Theorem 3.1 and Lemma 3.2.

Corollary 3.4 *Let $f(t)$ be a uniformly bounded continuous function on $(-\infty, \infty)$. Then for $x < 1/2$*

$$\sum_{k=0}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0). \tag{3.7}$$

For $x > 3 + 2\sqrt{2}$ or $x < 3 - 2\sqrt{2}$ and $x \neq -1$

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2} \right)^{k+1} \left(\delta^{2k} f(1) - x\delta^{2k} f(0) \right). \tag{3.8}$$

Finally, for $x > 3+2\sqrt{2}$ *or* $x < 3-2\sqrt{2}$ *and* $x \neq -1$ *and* $x \neq 0$, *we have*

$$\sum_{k=1}^{\infty} f(k)x^k = \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(x^{-1}\delta^{2k}f(0) - \delta^{2k}f(-1)\right). \qquad (3.9)$$

Proof. Taking limit $n \to \infty$ in Eqs. (3.4)-(3.6) yields Eqs. (3.7)-(3.9), respectively.

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