Convexity of Spherical Bernstein-Bézier Patches and Circular Bernstein-Bézier Curves

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Abstract

This paper discusses the criteria of convexity of spherical Bernstein-Bézier patches, circular Bernstein-Bézier curves, and homogeneous Bernstein-Bézier polynomials.

keywords. spherical Bernstein-Bézier patch, spherical Bernstein-Bézier polynomial, circular Bernstein-Bézier curve, homogeneous Bernstein-Bézier polynomial, Bézier coefficient, convexity.

1. Introduction

Let S be the unit sphere in R^3 with center at the origin. $T = \{v \in S : v = b_1v_1 + b_2v_2 + b_3v_3,\ b_i \geq 0\}$ is the spherical triangle generated by the three unit vectors $v_1,\ v_2,\ v_3 \in S$. Here the boundary of T, three circular arcs, lie on great circles. Let v be a point on S^1. The (spherical) barycentric coordinates of v relative to T are the unique real numbers b_1, b_2, and b_3 such that

$$v = b_1v_1 + b_2v_2 + b_3v_3. \tag{1}$$

It is clear from (1) that the spherical barycentric coordinates of a point v on the sphere S^1 are exactly the same as the trihedral coordinates of v with respect to the trihedron generated by $\{v_1, v_2, v_3\}$. This implies that they have the following properties (cf. [1]):

(i). At the vertices v_j, j=1, 2, 3, of T, $b_i(v_j) = \delta_{ij}$, i=1,2,3.

(ii). For all v in the interior of T, $b_i(v) > 0$.

(iii). In contrast to the usual barycentric coordinates on planar triangles (which always sum up to 1), $b_1(v) + b_2(v) + b_3(v) > 1$ if v$\in$T and v$\neq v_1, v_2, v_3$.

For the set f=$\{f_i : i = (i_1, i_2, i_3),\ i_1, i_2, i_3 \geq 0,\ |i| = i_1 + i_2 + i_3 = n\}$, an n^{th} degree functional spherical Bernstein-Bézier (SBB) polynomial is defined on the spherical triangle T as follows ([1]).

$$p_n(v) = B_n[f; b] = \sum_{|i|=n} f_i \phi_i^n(b), \tag{2}$$

where v_1, v_2, v_3 are three vertices of T, b=(b_1, b_2, b_3), v=$b_1 v_1 + b_2 v_2 + b_3 v_3$, and

$$\phi_i^n(b) = \frac{n!}{i!} b^i = \frac{n!}{i! i_2! i_3!} b_i^{i_1} b_2^{i_2} b_3^{i_3},\ |i| = i_1 + i_2 + i_3 = n. \tag{3}$$

f=$\{f_i\}$ is called the set of Bézier coefficients of the polynomial (2). If we do not restrict $\{v_1, v_2, v_3\}$ to be on the unit sphere S, then the $p_n(v)$ shown in (2) is called a homogeneous Bernstein-Bézier (HBB) polynomial of degree n on the trihedron $\hat{T} := \{v \in R^3 : v = b_1 v_1 + b_2 v_2 + b_3 v_3,\ b_i \geq 0\}$ generated by $\{v_1, v_2, v_3\}$ ([1]).

In many applications, the Bézier representation is used to form parametric surface patches by using vector-valued coefficients $\mathbf{f}$=$\{\mathbf{f}_i\}_{|i|=n}$. This will be indicated by using the boldface notation

$$\mathbf{p}_n(v) = \mathbf{B}_n[\mathbf{f}; b] = \sum_{|i|=n} \mathbf{f}_i \phi_i^n(b). \tag{4}$$

$\{(v^i, \mathbf{f}_i)\}_{|i|=n}$ is called the Bézier net of $\mathbf{p}_n(v)$. Here $v^i = \frac{1}{|i|} \sum_{\ell=1}^{3} i_\ell v_\ell$. In [1], the spherical Bernstein-Bézier (SBB) patch was defined as the surface $\{p_n(v)v : v \in T\}$. Using the notation $E_m^\ell c_i = c_{i+me^\ell}$, where e^ℓ is the ℓ^{th} coordinate vector in R^3, we can rewrite $p_n(v)v$ as

$$p_n(v)v = \sum_{|i|=n+1} \frac{1}{n+1} (i_1 v_1 E_{-1}^1 + i_2 v_2 E_{-1}^2 + i_3 v_3 E_{-1}^3) c_i \phi_i^{n+1}(b). \tag{5}$$

Clearly, from (5), $p_n(v)v$ is also a parametric surface patch $\mathbf{p}_{n+1}(v)$ with

$$\mathbf{f}_i = \frac{1}{|i|}(i_i v_i E^1_{-1} + i_2 v_2 E^2_{-1} + i_3 v_3 E^3_{-1})c_i, \ |i| = n+1. \tag{6}$$

For this reason we also called (4) the SBB patch of degree n defined on the spherical triangle T.

In [2], a theory of the circular Bernstein-Bézier (CBB) polynomials is developed. In addition to their intrinsic interest, the CBB polynomials are also useful for describing the behavior of SBB polynomials on the circular arcs making up the edges of spherical triangles.

Let C be the unit circle in R^2 with center at the origin, and let A be a circular arc on C with length less than π and vertices $v_1 \neq v_2$. Let v be a point on C. Then the (circular) barycentric coordinates of v relative to A are the unique pair of real numbers b_1, b_2 such that

$$v = b_1 v_1 + b_2 v_2. \tag{7}$$

Circular barycentric coordinates have a very simple form if we express points on C in polar coordinates. Suppose

$$v_1 = (\cos\theta_1, \sin\theta_1)^T, \qquad v_2 = (\cos\theta_2, \sin\theta_2)^T, \tag{8}$$

with $0 < \theta_2 - \theta_1 < \pi$. let $v \in C$ be expressed in polar coordinates as $v = (\cos\theta, \sin\theta)^T$. The circular barycentric coordinates of v relative to circular arc A are

$$b_1(v) = \frac{\sin(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)}, \qquad b_2(v) = \frac{\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}.$$

Similarly, for a given integer $n > 0$, the Bernstein basis polynomial of degree n on the circular arc A is

$$\phi_i^n(\theta) := \binom{n}{i} b_1(\theta)^{n-i} b_2(\theta)^i, \quad i = 0, 1, \cdots, n.$$

We call

$$p(\theta) := \sum_{i=0}^{n} c_i \phi_i^n(\theta) \tag{9}$$

a circular Bernstein-Bézier (CBB) polynomial of degree n on the circular arc A. Given a CBB polynomial p defined on a circular arc A, we define an associated CBB curve by

$$P(\theta) = p(\theta)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}. \tag{10}$$

The aim of this paper is to study the convexity properties of SBB patches and obtain some convexity criteria for HBB polynomials.

The paper is organized as follows. In Section 2, we will discuss the convexity of SBB patches $\mathbf{p}_n(v)$. Some properties of spherical barycentric coordinates will also be given in this section. In Section 3, we will discuss the convexity of CBB curves. Some convexity criteria for HBB polynomials will be shown in Section 4.

2. Convexity criteria of SBB patches

In order to discuss the convexity of SBB patches, we need the following lemmas about the relations among the spherical barycentric coordinates of v that are defined in equation (1).

Lemma 1. Let T be a spherical triangle with vertices $v_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, let $v = (x, y, z)$ be a point on S, the unit sphere in R^3 with the center at the origin, and let the vector $b = (b_1, b_2, b_3)$ be the spherical barycentric coordinates of v relative to T. Then

$$\sum_{1 \le i \ne j \le 3} b_i b_j \langle v_i, v_j \rangle = 1. \tag{11}$$

Proof. From equation (1), we have

$$\langle v, v\rangle = \langle b_1 v_1 + b_2 v_2 + b_3 v_3, b_1 v_1 + b_2 v_2 + b_3 v_3\rangle = \sum_{1\le i\ne j\le 3} b_i b_j \langle v_i, v_j\rangle.$$

Since $\langle v, v\rangle = 1$, we obtain equation (11).

Lemma 2. Let T be a spherical triangle with vertices $v_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, let $v = (x, y, z)$ be a point on S, the unit sphere in R^3 with the center at the origin, and let the vector $b = (b_1, b_2, b_3)$ be the spherical barycentric coordinates of v relative to T. Then b_3 can be considered as a function of b_1 and b_2, and

$$\frac{\partial b_3}{\partial b_i} = -\frac{\beta_i(b)}{\beta_3(b)},\ i = 1, 2, \tag{12}$$

where $\beta_\ell = \sum_{k=1}^{3} b_k\langle v_\ell, v_k\rangle$, $\ell = 1, 2, 3$, and $\langle v_\ell, V_k\rangle$ is the inner product of v_ℓ and v_k.

Proof. It is sufficient to prove the expression of $\frac{\partial b_3}{\partial b_1}$. Taking derivative in terms of b_1 on both sides of equation (11), we have

$$\frac{\partial}{\partial b_1}\sum_{1\le i,j\le 3} b_i b_j\langle v_i, v_j\rangle = 0.$$

Expanding the left hand side of the above equation and transposing all terms with $\frac{\partial b_3}{\partial b_1}$ to the right hand side, we obtain

$$b_1\langle v_1, v_1\rangle + b_2\langle v_1, v_2\rangle + b_3\langle v_1, v_3\rangle = -\frac{\partial b_3}{\partial b_1}[b_1\langle v_3, v_1\rangle + b_2\langle v_3, v_2\rangle + b_3\langle v_3, v_3\rangle].$$

Thus the lemma is proved.

Lemma 3. Let the SBB patch of f be the $\mathbf{B}_n[f, b]$ defined in (4). Then for $\ell = 1, 2$,

$$\frac{\partial}{\partial b_\ell}\mathbf{B}_n[f; b] = \frac{1}{\beta_3(b)}\sum_{|i|=n}\langle\psi(v, E), v_3 E_1^\ell - v_\ell E_1^3\rangle \mathbf{f}_i \phi_i^n(b), \tag{13}$$

where $\beta_3 = \sum_{k=1}^{3} b_k \langle v_3, v_k\rangle$, $\psi(v,E) = i_1 v_1 E^1_{-1} + i_2 v_2 E^2_{-1} + i_3 v_3 E^3_{-1}$, $\langle a v_\alpha\ E^\ell_m,\ b v_\beta\ E^q_n\rangle = ab\langle v_\alpha,\ v_\beta\rangle E^\ell_m E^q_n$, and the shift operator $E^\ell_m c_i = c_{i+me^\ell}$. Here e^ℓ is the ℓ^{th} coordinate vector in R^3.

Proof. It is sufficient to prove the expression $\frac{\partial}{\partial b_1}\mathbf{B}_n[f;b]$. Noting equation (12), we have

$$\frac{\partial}{\partial b_1}\mathbf{B}_n[f;b] = \sum_{|i|=n} \mathbf{f}_i \left[\frac{n!}{(i_1-1)!i_2!i_3!} b_1^{i_1-1} b_2^{i_2} b_3^{i_3} + \frac{n!}{i_1!i_2!(i_3-1)!} b_1^{i_1} b_2^{i_2} b_3^{i_3-1} \frac{\partial b_3}{\partial b_1}\right]$$

$$= \frac{1}{\beta_3(b)} \left[\sum_{|i|=n} (b_1\langle v_3, v_1\rangle + b_2\langle v_3, v_2\rangle + b_3\langle v_3, v_3\rangle)\mathbf{f}_i \frac{n!}{(i_1-1)!i_2!i_3!} b_1^{i_1-1} b_2^{i_2} b_3^{i_3}\right.$$

$$\left. - \sum_{|i|=n} (b_1\langle v_1, v_1\rangle + b_2\langle v_1, v_2\rangle + b_1\langle v_3, v_3\rangle)\mathbf{f}_i \frac{n!}{i_1!i_2!(i_3-1)!} b_1^{i_1} b_2^{i_2} b_3^{i_3-1}\right]$$

$$= \frac{1}{\beta_3(b)} \sum_{|i|=n} [i_1\langle v_3, v_1\rangle E^1_{-1}E^1_1 + i_2\langle v_3, v_2\rangle E^1_1 E^2_{-1} + i_3\langle v_3, v_3\rangle E^1_1 E^3_{-1}$$

$$- i_1\langle v_1, v_1\rangle E^1_{-1}E^3_1 - i_2\langle v_1, v_2\rangle E^2_{-1}E^3_1 - i_3\langle v_1, v_3\rangle E^3_1 E^3_{-1}]\mathbf{f}_i \phi^n_i(b)$$

$$= \frac{1}{\beta_3(b)} \sum_{|i|=n} (i_1 v_1 E^1_{-1} + i_2 v_2 E^2_{-1} + i_3 v_3 E^3_{-1}) \cdot (v_3 E^1_1 - v_1 E^3_1)\mathbf{f}_i \phi^n_i(b).$$

Thus equation (13) has been proved.

Similarly, we obtain

$$\frac{\partial^2}{\partial b_\ell^2}\mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} [\langle\psi(v,E), v_3 E^\ell_1 - v_\ell E^3_1\rangle]^2 \mathbf{f}_i \phi^n_i(b)$$
$$+ \beta_3(b)\frac{\partial}{\partial b_\ell}\mathbf{B}_n[f;b]\frac{\partial}{\partial b_\ell}\left(\frac{1}{\beta_3(b)}\right), \quad \ell = 1, 2, \tag{14}$$

and

$$\frac{\partial^2}{\partial b_1 \partial b_2}\mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \left[\prod_{\ell=1}^{2}\langle\psi(v,E), v_3 E^\ell_1 - v_\ell E^3_1\rangle\right] \mathbf{f}_i \phi^n_i(b)$$
$$+ \beta_3(b)\frac{\partial}{\partial b_2}\mathbf{B}_n[f;b]\frac{\partial}{\partial b_1}\left(\frac{1}{\beta_3(b)}\right),$$

$$= \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \left[\prod_{\ell=1}^{2} \langle \psi(v,E), v_3 E_1^\ell - v_\ell E_1^3 \rangle \right] \mathbf{f}_i \phi_i^n(b)$$

$$+ \beta_3(b) \frac{\partial}{\partial b_1} \mathbf{B}_n[f;b] \frac{\partial}{\partial b_2} \left(\frac{1}{\beta_3(b)} \right). \quad (15)$$

We now denote

$$\mathbf{p}_i = \langle \psi(v,E), V_3 E_1^1 - V_1 E_1^3 \rangle \mathbf{f}_i, \quad (16)$$

$$\mathbf{q}_i = \langle \psi(v,E), v_3 E_1^2 - v_2 E_1^3 \rangle \mathbf{f}_i, \quad (17)$$

$$\mathbf{U}_i = \langle \psi(v,E), v_1 E_1^3 - v_3 E_1^1 \rangle \langle v_3 E_1^2 - v_3 E_1^1 + (v_1 - v_2) E_1^3, \psi(v,E) \rangle \mathbf{f}_i, \quad (18)$$

$$\mathbf{V}_i = \langle \psi(v,E), v_2 E_1^3 - v_3 E_1^2 \rangle \langle v_3 E_1^1 - v_3 E_1^2 + (v_2 - v_1) E_1^3, \psi(v,E) \rangle \mathbf{f}_i, \quad (19)$$

and

$$\mathbf{W}_i = \langle \psi(v,E), v_3 E_1^1 - v_1 E_1^3 \rangle \langle v_3 E_1^2 - v_2 E_1^3, \psi(v,E) \rangle \mathbf{f}_i. \quad (20)$$

Remark: By changing $\mathbf{f}_i$ to f_i in (13)-(20), we obtain the corresponding partial derivatives of $B_n[f;b]$ and the corresponding p_i, q_i, U_i, V_i, and W_i.

Obviously, we have

$$\frac{\partial}{\partial b_1} \mathbf{B}_n[f;b] = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{p}_i \phi_i^n(b),$$

$$\frac{\partial}{\partial b_2} \mathbf{B}_n[f;b] = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{q}_i \phi_i^n(b),$$

$$\frac{\partial^2}{\partial b_1^2} \mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} (\mathbf{U}_i + \mathbf{W}_i) \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_1} \mathbf{B}_n \frac{\partial}{\partial b_1} \left(\frac{1}{\beta_3(b)} \right),$$

$$\frac{\partial^2}{\partial b_2^2} \mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} (\mathbf{V}_i + \mathbf{W}_i) \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_2} \mathbf{B}_n \frac{\partial}{\partial b_2} \left(\frac{1}{\beta_3(b)} \right),$$

and

$$\frac{\partial^2}{\partial b_1^2 b_2} \mathbf{B}_n[f;b] = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{W}_i \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_1} \mathbf{B}_n \frac{\partial}{\partial b_2} \left(\frac{1}{\beta_3(b)} \right),$$

$$= \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{W}_i \phi_i^n(b) + \beta_3(b) \frac{\partial}{\partial b_2} \mathbf{B}_n \frac{\partial}{\partial b_1} \left(\frac{1}{\beta_3(b)} \right).$$

It is well known that if the Gaussian curvature of a compact surface π in R^3 is positive everywhere, then surface π is convex (lies on one side of each tangent plane)([9]). In addition, if π is defined by $\mathbf{r} = \mathbf{r}(u, v)$, a parametric vector-valued function in C^2, then the Gaussian curvature of π is $k = (LN - M^2)/(EG - F^2)$, where E, G, F and L, M, N are respectively the first fundamental form and the second fundamental form of π. It is well known that $L = \frac{1}{D}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uu})$, $M = \frac{1}{D}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uv})$, $N = \frac{1}{D}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{vv})$, and $D^2 = EG - F^2 > 0$. Here $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the scalar product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and is defined by $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \rangle$. Therefore the Gaussian curvature K of $\mathbf{r} = \mathbf{r}(u, v)$ and $LN - M^2$, or $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uu})$, $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{vv}) - (\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uv})^2$ have the same sign. In particular, for the surface $\mathbf{r} = \mathbf{r}(b_1, b_2) = \mathbf{B}_n[f; b]$, we have

$$\mathbf{r}_{b_1} = \frac{\partial}{\partial b_1} \mathbf{B}_n[f; b], \quad \mathbf{r}_{b_2} = \frac{\partial}{\partial b_2} \mathbf{B}_n[f; b], \quad \mathbf{r}_{b_1 b_1} = \frac{\partial^2}{\partial b_1^2} \mathbf{B}_n[f; b],$$

etc. Thus, if $(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_1})$ $(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_2 b_2})$ $-(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_2})^2 > 0$, then $K > 0$. By using the notation in (16)-(20), from Lemma 3, we have

$$\begin{aligned}&(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_1})(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_2 b_2}) - (\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_2})^2 \\ &= (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^U)(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^V) + (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^U)(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^W) + (\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^V)(\mathbf{r}^p, \mathbf{r}^q, \mathbf{r}^W),\end{aligned} \quad (21)$$

where $\mathbf{r}^p = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{p}_i \phi_i^n(b)$, $\mathbf{r}^q = \frac{1}{\beta_3(b)} \sum_{|i|=n} \mathbf{q}_i \phi_i^n(b)$, $\mathbf{r}^U = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{U}_i \phi_i^n(b)$, $\mathbf{r}^V = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{V}_i \phi_i^n(b)$, and $\mathbf{r}^W = \frac{1}{[\beta_3(b)]^2} \sum_{|i|=n} \mathbf{W}_i \phi_i^n(b)$.

Denoting $\nabla_{i,j,k}^{(1)} = (\mathbf{p}_i, \mathbf{q}_j, \mathbf{U}_k)$, $\nabla_{i,j,k}^{(2)} = (\mathbf{p}_i, \mathbf{q}_j, \mathbf{V}_k)$, and $\nabla_{i,j,k}^{(3)} = (\mathbf{p}_i, \mathbf{q}_j, \mathbf{W}_k)$, from (21) we have the following theorem.

Theorem 1. If

$$\sum_{1 \le \alpha < \beta \le 3} \sum_{|j|=3n} \sum_{|s|=2n} \sum_{|r|=2n} \sum_{|t|=n} \sum_{|k|=n} \triangle_{j,s,r,t,k}^{(\alpha,\beta)}(i) > 0 \quad (22)$$

holds for all $i \in Z_+^3$, $|i| = 6n$, where

$$\triangle_{j,s,r,t,k}^{(\alpha,\beta)}(i) = \nabla_{t,s-t,j-s}^{(\alpha)} \nabla_{k,r-k,i-j-r}^{(\beta)} \frac{\binom{s}{t}\binom{r}{k}\binom{j}{s}\binom{i-j}{r}\binom{i}{j}}{\binom{2n}{n}^2\binom{3n}{2n}^2\binom{6n}{3n}},$$

then $\mathbf{B}_n[f;b]$ is convex over the spherical triangle T.

Proof. To prove the theorem, we need the following two representations.

$$\sum_{|i|=n} \mathbf{f}_i \phi_i^n(b) \times \sum_{|i|=m} \mathbf{q}_i \phi_i^m(b) = \sum_{|i|=n+m} \mathbf{h}_i \phi_i^{n+m}(b), \tag{23}$$

where $\mathbf{h}_i = \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{q}_{i-j}) \binom{i}{j} / \binom{n+m}{n}$, and

$$\sum_{|i|=n} f_i \phi_i^n(b) \cdot \sum_{|i|=m} g_i \phi_i^m(b) = \sum_{|i|=n+m} h_i \phi_i^{n+m}(b), \tag{24}$$

where $h_i = \sum_{|j|=n} (f_j g_{i-j}) \binom{i}{j} / \binom{n+m}{n}$. In fact,

$$\begin{aligned}
&\sum_{|j|=n} \mathbf{f}_j \phi_j^n(b) \times \sum_{|i|=m} \mathbf{g}_i \phi_i^m(b) \\
&= \sum_{|i|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_i)\, \phi_j^n(b) \phi_i^m(b) \\
&= \sum_{|i|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_i) \binom{n}{j} \binom{m}{i} b^{i+j} \\
&= \sum_{|i'|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_{i'-j}) \binom{n}{j} \binom{m}{i'-j} b^{i'} \\
&= \sum_{|i'|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_{i'-j}) \frac{n!}{j!} \frac{m!}{(i'-j)!} b^{i'} \\
&= \sum_{|i'|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_{i'-j}) \frac{i!}{(i'-j)!j!} \frac{(n+m)!}{i'!} \frac{n!m!}{(n+m)!} b^{i'} \\
&= \sum_{|i'|=m} \sum_{|j|=n} (\mathbf{f}_j \times \mathbf{g}_{i'-j}) \binom{i'}{j} \phi_{i'}^{n+m}(b) / \binom{n+m}{n}.
\end{aligned}$$

Similarly, we may obtain equation (24) by using the same argument.

Therefore,

$$\begin{aligned}(\mathbf{r}^p,\mathbf{r}^q,\mathbf{r}^U) =&(\mathbf{r}^p\times\mathbf{r}^q)\cdot\mathbf{r}^U\\ =&\left(\frac{1}{\beta_3(b)}\sum_{|t|=n}\mathbf{p}_t\phi_t^n(b)\times\frac{1}{\beta_3(b)}\sum_{|s|=n}\mathbf{q}_s\phi_s^n(b)\right)\cdot\frac{1}{[\beta_3(b)]^2}\sum_{|i|=n}\mathbf{U}_i\phi_i^n(b)\\ =&\frac{1}{[\beta_3(b)]^4}\sum_{|s|=2n}\left(\sum_{|t|=n}\mathbf{p}_t\times\mathbf{q}_{s-t}\right)\frac{\binom{s}{t}}{\binom{2n}{n}}\phi_s^{2n}(b)\cdot\sum_{|i|=n}\mathbf{U}_i\phi_i^n(b)\\ =&\frac{1}{[\beta_3(b)]^4}\sum_{|i|=3n}\sum_{|s|=2n}\sum_{|t|=n}(\mathbf{p}_t\times\mathbf{q}_{s-t})\cdot\mathbf{U}_{i-s}\binom{i}{s}\binom{s}{t}\phi_i^{3n}(b)/\binom{2n}{n}\binom{3n}{2n}\\ =&\frac{1}{[\beta_3(b)]^4}\sum_{|i|=3n}\sum_{|s|=2n}\sum_{|t|=n}\nabla^{(1)}_{t,s-t,i-s}\binom{i}{s}\binom{s}{t}\phi_i^{3n}(b)/\binom{2n}{n}\binom{3n}{2n}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\mathbf{r}^p,\mathbf{r}^q,\mathbf{r}^V) =&(\mathbf{r}^p\times\mathbf{r}^q)\cdot\mathbf{r}^V\\ =&\left(\frac{1}{\beta_3(b)}\sum_{|k|=n}\mathbf{p}_k\phi_k^n(b)\times\frac{1}{\beta_3(b)}\sum_{|r|=n}\mathbf{q}_r\phi_r^n(b)\right)\cdot\frac{1}{[\beta_3(b)]^2}\sum_{|i|=n}\mathbf{V}_i\phi_i^n(b)\\ =&\frac{1}{[\beta_3(b)]^4}\sum_{|i|=3n}\sum_{|r|=2n}\sum_{|k|=n}\nabla^{(2)}_{k,r-k,i-r}\binom{i}{r}\binom{r}{k}\phi_i^{3n}(b)/\binom{2n}{n}\binom{3n}{2n}.\end{aligned}$$

Thus,

$$\begin{aligned}&(\mathbf{r}^p,\mathbf{r}^q,\mathbf{r}^U)\cdot(\mathbf{r}^p,\mathbf{r}^q,\mathbf{r}^V)\\ =&\frac{1}{[\beta_3(b)]^4}\sum_{|i|=3n}\sum_{|s|=2n}\sum_{|t|=n}\nabla^{(1)}_{t,s-t,i-s}\binom{i}{s}\binom{s}{t}\phi_i^{3n}(b)\\ &\cdot\frac{1}{[\beta_3(b)]^4}\sum_{|i|=3n}\sum_{|r|=2n}\sum_{|k|=n}\nabla^{(2)}_{k,r-k,i-r}\binom{i}{r}\binom{r}{k}\phi_i^{3n}(b)/\binom{2n}{n}^2\binom{3n}{2n}^2\\ =&\frac{1}{[\beta_3(b)]^8}\sum_{|i|=6n}\sum_{|j|=3n}\sum_{|s|=2n}\sum_{|r|=2n}\sum_{|t|=n}\sum_{|k|=n}\\ &\nabla^{(1)}_{t,s-t,i-s}\nabla^{(2)}_{k,r-k,i-j-r}\frac{\binom{s}{t}\binom{r}{k}\binom{j}{s}\binom{i-j}{r}\binom{i}{j}}{\binom{2n}{n}^2\binom{3n}{2n}^2\binom{6n}{3n}}\phi_i^{6n}(b).\end{aligned}$$

By using the above two representations and expression (17), we immediately have

$$(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_1})(\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_2 b_2}) - (\mathbf{r}_{b_1}, \mathbf{r}_{b_2}, \mathbf{r}_{b_1 b_2})^2$$
$$= \frac{1}{[\beta_3(b)]^4} \sum_{|i|=6n} \sum_{|j|=3n} \sum_{|s|=2n} \sum_{|r|=2n} \sum_{|t|=n} \sum_{|k|=n} \sum_{1 \le \alpha < \beta \le 3} \triangle^{(\alpha,\beta)}_{j,s,r,t,k}(i) \phi^{6n}_i(b). \quad (25)$$

Thus Theorem 1 is proved.

From condition (25), we also have the following criterion for the convexity of $\mathbf{B}_n[f; b]$.

Theorem 2. If one of the following condition is satisfied for all $i, j, k \in Z^3_+$, $|i| = |j| = |k| = n$, then $\mathbf{B}_n[f; b]$ is convex over the spherical triangle T.

(i) $\nabla^{(u)}_{i,j,k} + \nabla^{(w)}_{i,j,k} > 0, \quad \nabla^{(u)}_{i,j,k}\nabla^{(v)}_{i,j,k} + \nabla^{(v)}_{i,j,k}\nabla^{(w)}_{i,j,k} + \nabla^{(w)}_{i,j,k}\nabla^{(u)}_{i,j,k} > 0;$

(ii) $\nabla^{(1)}_{i,j,k} + \nabla^{(3)}_{i,j,k} > \left|\nabla^{(3)}_{i,j,k}\right|$ *and* $\nabla^{(2)}_{i,j,k} + \nabla^{(3)}_{i,j,k} > \left|\nabla^{(3)}_{i,j,k}\right|;$

(iii) $\sum_{1 \le \alpha < \beta \le 3} \nabla^{(\alpha)}_{i,j,k}\nabla^{(\beta)}_{i,j,k} > 0;$

(iv) $\nabla^{(\alpha)}_{i,j,k} > 0, \quad \alpha = 1, 2, 3,$

where (u, v, w) is a permutation of (1, 2, 3).

Proof. Denote $\nabla^{(\ell)}_{i,j,k}$, $\ell = 1, 2, 3$, by a_1, b_1, c_1, respectively and $\nabla^{(\ell)}_{i',j',k'}$, $\ell = 1, 2, 3$, by a_2, b_2, c_2, respectively. It is obvious that inequality (22) holds if

$$a_1 b_2 + b_1 c_2 + c_1 a_2 + a_2 b_1 + b_2 c_1 + c_2 a_1 > 0 \quad (26)$$

for all $i, j, k \in Z^3_+$, where $|i| = |j| = |k| = n$ and $|i'| = |j'| = |k'| = n$.

We can prove that inequality (26) holds if the following inequalities

$$a_\ell + c_\ell > 0, \; and \; a_\ell b_\ell + b_\ell c_\ell + c_\ell a_\ell > 0, \qquad \ell = 1, 2,$$

hold. In fact, if the above inequalities hold, then we have

$$(a_1 + c_1) b_1 > -a_1 c_1, \qquad (a_2 + c_2) b_2 > -a_2 c_2,$$

and

$$
\begin{aligned}
&(a_1+c_1)(a_2+c_2)(a_1b_2+b_1c_2+c_1a_2+a_2b_1+b_2c_1+c_2a_1)\\
=&(a_1+c_1)(a_2+c_2)[b_1(a_2+c_2)+b_2(a_1+c_1)+c_1a_2+a_1c_2]\\
=&(a_2+c_2)^2(a_1+c_1)b_1+(a_1+c_1)^2(a_2+c_2)b_2+c_1a_2+a_1c_2\\
>&-a_1c_1(a_2+c_2)^2-a_2c_2(a_1+c_1)^2+c_1a_2+a_1c_2\\
=&(a_1c_2-a_2c_1)^2\geq 0.
\end{aligned}
$$

Thus inequality (26) holds.

Consequently, inequality (22) holds if

$$\nabla^{(1)}_{i,j,k}+\nabla^{(3)}_{i,j,k}>0,\ \textit{and}\ \nabla^{(1)}_{i,j,k}\nabla^{(2)}_{i,j,k}+\nabla^{(2)}_{i,j,k}\nabla^{(3)}_{i,j,k}+\nabla^{(3)}_{i,j,k}\nabla^{(1)}_{i,j,k}>0,$$

which is equivalent to that the following matrix is positive definite.

$$\begin{bmatrix}\nabla^{(1)}_{i,j,k}+\nabla^{(3)}_{i,j,k} & \nabla^{(1)}_{i,j,k}\\ \nabla^{(1)}_{i,j,k} & \nabla^{(1)}_{i,j,k}+\nabla^{(2)}_{i,j,k}\end{bmatrix}.$$

Obviously, the above matrix is positive definite if it is strictly strongly diagonally dominant, that is,

$$\nabla^{(1)}_{i,j,k}+\nabla^{(3)}_{i,j,k}>\left|\nabla^{(1)}_{i,j,k}\right|\ \textit{and}\ \nabla^{(1)}_{i,j,k}+\nabla^{(2)}_{i,j,k}>\left|\nabla^{(1)}_{i,j,k}\right|.$$

Furthermore, this condition can be implied by

$$\nabla^{(\alpha)}_{i,j,k}>0,\quad \alpha=1,2,3.$$

Thus, Theorem 2 is proved.

Remark 5. Equation (23) and inequality (26) were given in [12] without any proof.

3. Convexity criteria of CBB curvess

In this section, we will give the convexity criteria for CBB curves. Obviously, a CBB curve is convex if and only if its curvature $k \geq 0$; i.e., the curve lies on only one side of each tangent line. The following lemma gives the curvature $k = k(\theta)$ of $P(\theta)$ at θ, which is defined as equation (10).

Lemma 4. Let the $P(\theta)$ defined as (10) be a CBB curve and $p(\theta)$ be the associated CBB polynomial defined in equation (9). The CBB curve is convex if and only if

$$(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta) \geq 0. \tag{27}$$

Proof. It is sufficient to prove that the sign of the curvature of the CBB curve $P(\theta)$ at any θ is

$$Sign[k(\theta)] = Sign[(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta)]. \tag{28}$$

Obviously, the curvature of the parametric curve $P(\theta) = (x(\theta), y(\theta))$ is

$$k(\theta) = \frac{y''(\theta)x'(\theta) - x''(\theta)y'(\theta)}{[(x'(\theta))^2 + (y'(\theta))^2]^{3/2}}.$$

Since $x(\theta) = p(\theta)\cos\theta$ and $y(\theta) = p(\theta)\sin\theta$, we obtain

$$\begin{aligned}
&y''(\theta)x'(\theta) - x''(\theta)y'(\theta)\\
=&[p''(\theta)\sin\theta + 2p'(\theta)\cos\theta - p(\theta)\sin\theta][p'(\theta)\cos\theta - p(\theta)\sin\theta]\\
&- [p''(\theta)\cos\theta - 2p'(\theta)\sin\theta - p(\theta)\cos\theta][p'(\theta)\sin\theta + p(\theta)\cos\theta]\\
=&(p(\theta))^2(\sin^2\theta + \cos^2\theta) + 2(p'(\theta))^2(\sin^2\theta + \cos^2\theta) - p(\theta)p''(\theta)(\sin^2\theta + \cos^2\theta)\\
=&(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta).
\end{aligned}$$

Thus the lemma is proved.

We now derive $p'(\theta)$ and $p''(\theta)$. Since $p(\theta) = \sum_{i=0}^{n} c_i \binom{n}{i} b_1(\theta)^{n-i} b_2(\theta)^i$, where $b_1(\theta) = \sin(\theta_2 - \theta)/\sin(\theta_2 - \theta_1)$ and $b_2(\theta) = \sin(\theta - \theta_1)/\sin(\theta_2 - \theta_1)$, we have

$$\begin{aligned}
p'(\theta) = &- \sum_{i=0}^{n-1} c_i \frac{n!}{(n-i-1)!i!} b_1(\theta)^{n-i-1} b_2(\theta)^i \frac{\cos(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)} \\
&+ \sum_{i=1}^{n} c_i \frac{n!}{(n-i)!(i-1)!} b_1(\theta)^{n-i} b_2(\theta)^{i-1} \frac{\cos(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)} \\
= &- n \sum_{i=0}^{n-1} c_i \binom{n-1}{i} b_1(\theta)^{n-i-1} b_2(\theta)^i \frac{\cos(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)} \\
&+ n \sum_{i=0}^{n-1} c_{i+1} \binom{n-1}{i} b_1(\theta)^{n-i-1} b_2(\theta)^i \frac{\cos(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)} \\
= &\frac{n}{\sin(\theta_2 - \theta_1)} \sum_{i=0}^{n-1} \left[\cos(\theta - \theta_1) c_{i+1} - \cos(\theta_2 - \theta) c_i\right] \phi_i^{n-1}(\theta).
\end{aligned}$$

Thus

$$\begin{aligned}
p''(\theta) = &\frac{n(n-1)}{\sin^2(\theta_2 - \theta_1)} \sum_{i=0}^{n-2} \left[(\cos(\theta - \theta_1) c_{i+2} - \cos(\theta_2 - \theta) c_{i+1}) \cos(\theta - \theta_1)\right. \\
&\left. - (\cos(\theta - \theta_1) c_{i+1} - \cos(\theta_2 - \theta) c_i) \cos(\theta_2 - \theta)\right] \phi_i^{n-2}(\theta) \\
&- \frac{n}{\sin(\theta_2 - \theta_1)} \sum_{i=0}^{n-1} \left[\sin(\theta - \theta_1) c_{i+1} + \sin(\theta_2 - \theta) c_i\right] \phi_i^{n-1}(\theta) \\
= &\frac{n(n-1)}{\sin^2(\theta_2 - \theta_1)} \sum_{i=0}^{n-2} \left[\cos^2(\theta - \theta_1) c_{i+2} - 2\cos(\theta_2 - \theta)\cos(\theta - \theta_1) c_{i+1}\right. \\
&\left. + \cos^2(\theta_2 - \theta) c_i\right] \phi_i^{n-2}(\theta) \\
&- n \sum_{i=0}^{n-1} \left[b_2(\theta) c_{i+1} + b_1(\theta) c_i\right] \frac{(n-1)!}{i!(n-i-1)!} b_1(\theta)^{n-i-1} b_2(\theta)^i \\
= &\frac{n(n-1)}{\sin^2(\theta_2 - \theta_1)} \sum_{i=0}^{n-2} \left[\cos^2(\theta - \theta_1) c_{i+2} - 2\cos(\theta_2 - \theta)\cos(\theta - \theta_1) c_{i+1}\right. \\
&\left. + \cos^2(\theta_2 - \theta) c_i\right] \phi_i^{n-2}(\theta) \\
&- \sum_{i=0}^{n-1} (i+1) c_{i+1} \phi_{i+1}^{n}(\theta) - \sum_{i=0}^{n-1} (n-i) c_i \phi_i^{n}(\theta)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right.\\
&\left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta)-\sum_{i=1}^{n} ic_i\phi_i^n(\theta)-\sum_{i=0}^{n-1}(n-i)c_i\phi_i^n(\theta)\\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right.\\
&\left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta)-\sum_{i=0}^{n} ic_i\phi_i^n(\theta)-\sum_{i=0}^{n}(n-i)c_i\phi_i^n(\theta)\\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[\cos^2(\theta-\theta_1)c_{i+2}-2\cos(\theta_2-\theta)\cos(\theta-\theta_1)c_{i+1}\right.\\
&\left.+\cos^2(\theta_2-\theta)c_i\right]\phi_i^{n-2}(\theta)-n\sum_{i=0}^{n} c_i\phi_i^n(\theta)
\end{aligned}
$$

Noting the trigonometric identities $\cos^2(\theta_2-\theta)=1-\sin^2(\theta_2-\theta)$, $\cos^2(\theta-\theta_1)=1-\sin^2(\theta-\theta_1)$, and $\cos(\theta_2-\theta)\cos(\theta-\theta_1)=\cos(\theta_2-\theta_1)+\sin(\theta_2-\theta)\sin(\theta-\theta_1)$, we may re-write $p''(\theta)$ into

$$
\begin{aligned}
p''(\theta)=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta)\\
&-n(n-1)\sum_{i=0}^{n-2} b_2(\theta)^2c_{i+2}\phi_i^{n-2}(\theta)-2n(n-1)\sum_{i=0}^{n-2} b_2(\theta)b_1(\theta)c_{i+1}\phi_i^{n-2}(\theta)\\
&-n(n-1)\sum_{i=0}^{n-2} b_1(\theta)^2c_i\phi_i^{n-2}(\theta)-n\sum_{i=0}^{n} c_i\phi_i^n(\theta)\\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{n-2}\left[c_{i+2}-2\cos(\theta_2-\theta_1)c_{i+1}+c_i\right]\phi_i^{n-2}(\theta)\\
&-n(n-1)\sum_{i=0}^{n-2} c_{i+2}\frac{(n-2)!}{(n-i-2)!i!}b_1(\theta)^{n-i-2}b_2(\theta)^{i+2}\\
&-2n(n-1)\sum_{i=0}^{n-2} c_{i+1}\frac{(n-2)!}{(n-i-2)!i!}b_1(\theta)^{n-i-1}b_2(\theta)^{i+1}
\end{aligned}
$$

$$
\begin{aligned}
&- n(n-1)\sum_{i=0}^{n-2} c_i \frac{(n-2)!}{(n-i-2)!i!} b_1(\theta)^{n-i} b_2(\theta)^i - n\sum_{i=0}^{n} c_i \phi_i^n(\theta) \\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{n-2} \left[c_{i+2} - 2\cos(\theta_2-\theta_1)c_{i+1} + c_i\right] \phi_i^{n-2}(\theta) \\
&- \sum_{i=0}^{n-2} (i+2)(i+1)c_{i+2} \frac{n!}{(n-i-2)!(i+2)!} b_1(\theta)^{n-i-2} b_2(\theta)^{i+2} \\
&- 2\sum_{i=0}^{n-2} (i+1)(n-i-1)c_{i+1} \frac{n!}{(n-i-1)!(i+1)!} b_1(\theta)^{n-i-1} b_2(\theta)^{i+1} \\
&- \sum_{i=0}^{n-2} (n-i)(n-i-1)c_i \frac{n!}{(n-i)!i!} b_1(\theta)^{n-i} b_2(\theta)^i - n\sum_{i=0}^{n} c_i \phi_i^n(\theta) \\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{n-2} \left[c_{i+2} - 2\cos(\theta_2-\theta_1)c_{i+1} + c_i\right] \phi_i^{n-2}(\theta) \\
&- \sum_{i=0}^{n} i(i-1)c_i\phi_i^n(\theta) - 2\sum_{i=0}^{n} i(n-i)c_i\phi_i^n(\theta) \\
&- \sum_{i=0}^{n} (n-i)(n-i-1)c_i\phi_i^n(\theta) - n\sum_{i=0}^{n} c_i\phi_i^n(\theta) \\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{n-2} \left[c_{i+2} - 2\cos(\theta_2-\theta_1)c_{i+1} + c_i\right] \phi_i^{n-2}(\theta) \\
&- \sum_{i=0}^{n} \left[i(i-1) + 2i(n-i) + (n-i)(n-i-1) + n\right] c_i\phi_i^n(\theta) \\
=&\frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{n-2} \left[c_{i+2} - 2\cos(\theta_2-\theta_1)c_{i+1} + c_i\right] \phi_i^{n-2}(\theta) \\
&- n^2\sum_{i=0}^{n} c_i\phi_i^n(\theta) \qquad (29)
\end{aligned}
$$

By using equation (24) for the case of two dimension:

$$
\sum_{i=0}^{n} f_i\phi_i^n(\theta) \cdot \sum_{j=0}^{m} g_j\phi_j^m(\theta)
$$

$$= \sum_{i=0}^{n+m} \sum_{j=0}^{m} f_{i-j} g_j \binom{n+m-i}{m-j} \phi_i^{n+m}(\theta) / \binom{n+m}{m}, \tag{30}$$

we obtain

$$[p(\theta)]^2 = \frac{1}{\binom{2n}{n}} \sum_{i=0}^{2n} d_i \phi_i^{2n}(\theta),$$

where

$$d_i = \sum_{j=0}^{n} c_j c_{i-j} \binom{i}{j} \binom{2n-i}{n-j}$$

and

$$\begin{aligned} p(\theta)p''(\theta) =& \frac{n(n-1)}{\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2} \sum_{j=0}^{n} c_j \left[c_{i-j+2} - 2\cos(\theta_2-\theta_1) c_{i-j+1}\right. \\ & \left.+ c_{i-j}\right] \binom{i}{j} \binom{2n-i-2}{n-j} \phi_i^{2n-2}(\theta) / \binom{2n-2}{n} \\ & - n^2 \sum_{i=0}^{2n} d_i \phi_i^{2n}(\theta) / \binom{2n}{n} \\ =& \frac{2n(2n-1)}{\binom{2n}{n} \sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2} \sum_{j=0}^{n} c_j \left[c_{i-j+2} - 2\cos(\theta_2-\theta_1) c_{i-j+1}\right. \\ & \left.+ c_{i-j}\right] \binom{i}{j} \binom{2n-i-2}{n-j} \phi_i^{2n-2}(\theta) \\ & - \frac{n^2}{\binom{2n}{n}} \sum_{i=0}^{2n} \sum_{j=0}^{n} c_j c_{i-j} \binom{i}{j} \binom{2n-i}{n-j} \phi_i^{2n}(\theta). \end{aligned}$$

Here, $c_{i-j} = 0$ if $i < j$ or $i > j + n$.

For the sake of convenience, we will use the relation $[(p(\theta))^2]'' = 2(p'(\theta))^2 + 2p(\theta)p''(\theta)$ to derive the expression for $(p'(\theta))^2$. Similar to the process of deriving equation (29), we can obtain

$$[(p(\theta))^2]'' = \frac{2n(2n-1)}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2} \left[d_{i+2} - 2\cos(\theta_2-\theta_1) d_{i+1} + d_i\right] \phi_i^{2n-2}(\theta)$$

$$
\begin{aligned}
&- \frac{(2n)^2}{\binom{2n}{n}} \sum_{i=0}^{2n} d_i \phi_i^n(\theta) \\
=&\frac{2n(2n-1)}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2}\sum_{j=0}^{n} c_j \left[c_{i-j+2}\binom{i+2}{j}\binom{2n-i-2}{n-j} \right. \\
&- 2\cos(\theta_2-\theta_1) c_{i-j+1}\binom{i+1}{j}\binom{2n-i-1}{n-j} \\
&\left. + c_{i-j}\binom{i}{j}\binom{2n-i}{n-j} \right] \phi_i^{2n-2}(\theta) \\
&- \frac{4n^2}{\binom{2n}{n}} \sum_{i=0}^{2n}\sum_{j=0}^{n} c_j c_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\phi_i^{2n}(\theta).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta) \\
=&(p(\theta))^2 + [(p(\theta))^2]'' - 3p(\theta)p''(\theta) \\
=&\frac{2n(2n-1)}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)} \sum_{i=0}^{2n-2}\sum_{j=0}^{n} c_j \left[c_{i-j+2}\left(\binom{i+2}{j}\binom{2n-i-2}{n-j} - 3\binom{i}{j}\binom{2n-i-2}{n-j}\right) \right. \\
&- 2\cos(\theta_2-\theta_1) c_{i-j+1}\left(\binom{i+1}{j}\binom{2n-i-1}{n-j} - 3\binom{i}{j}\binom{2n-i-2}{n-j}\right) \\
&\left. + c_{i-j}\left(\binom{i}{j}\binom{2n-i}{n-j} - 3\binom{i}{j}\binom{2n-i-2}{n-j}\right) \right] \phi_i^{2n-2}(\theta) \\
&+ \frac{1-n^2}{\binom{2n}{n}} \sum_{i=0}^{2n}\sum_{j=0}^{n} c_j c_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\phi_i^{2n}(\theta) \qquad (31)
\end{aligned}
$$

We will use the following degree-raising formula which was given in [2], to raise the degree of the first summation in the above expression of $(p(\theta))^2 + 2(p'(\theta))^2 - p(\theta)p''(\theta)$ to 2n.

$$
p(\theta) = \sum_{i=0}^{d} e_i \phi_i^d(\theta) = \sum_{i=0}^{d+2} \bar{e}_i \phi_i^{d+2}(\theta),
$$

where

$$
\bar{e}_i(\theta) = \frac{1}{(d+2)(d+1)} \left[i(i-1)e_{i-2} + 2\cos(\theta_2-\theta_1) i(d-i+2) e_{i-1} \right.
$$

$$+(d-i+2)(d-i+1)e_i]\,,$$

for $i=0,1,\cdots,d+2$.

Thus, from equation (31) and the degree-raising formula, we obtain

$$
\begin{aligned}
&(p(\theta))^2+2(p'(\theta))^2-p(\theta)p''(\theta)\\
=&\frac{1}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n}\sum_{j=0}^{n}c_j\left[i(i-1)\left[c_{i-j}\binom{2n-i}{n-j}\left(\binom{i}{j}-3\binom{i-2}{j}\right)\right.\right.\\
&-2\cos(\theta_2-\theta_1)c_{i-j-1}\left(\binom{i-1}{j}\binom{2n-i+1}{n-j}-3\binom{i-2}{j}\binom{2n-i}{n-j}\right)\\
&\left.+c_{i-j-2}\binom{i-2}{j}\left(\binom{2n-i+2}{n-j}-3\binom{2n-i}{n-j}\right)\right]\\
&+2\cos(\theta_2-\theta_1)i(2n-i)\left[c_{i-j+1}\binom{2n-i-1}{n-j}\left(\binom{i+1}{j}-3\binom{i-1}{j}\right)\right.\\
&-2\cos(\theta_2-\theta_1)c_{i-j}\left(\binom{i}{j}\binom{2n-i}{n-j}-3\binom{i-1}{j}\binom{2n-i-1}{n-j}\right)\\
&\left.+c_{i-j-1}\binom{i-1}{j}\left(\binom{2n-i+1}{n-j}-3\binom{2n-i-1}{n-j}\right)\right]\\
&+(2n-i)(2n-i-1)\left[c_{i-j+2}\binom{2n-i-2}{n-j}\left(\binom{i+2}{j}-3\binom{i}{j}\right)\right.\\
&-2\cos(\theta_2-\theta_1)c_{i-j+1}\left(\binom{i+1}{j}\binom{2n-i-1}{n-j}-3\binom{i}{j}\binom{2n-i-2}{n-j}\right)\\
&\left.\left.+c_{i-j}\binom{i}{j}\left(\binom{2n-i}{n-j}-3\binom{2n-i-2}{n-j}\right)\right]\right]\phi_i^{2n}(\theta)\\
&+\frac{1-n^2}{\binom{2n}{n}}\sum_{i=0}^{2n}\sum_{j=0}^{n}c_jc_{i-j}\binom{i}{j}\binom{2n-i}{n-j}\phi_i^{2n}(\theta)\\
=&\frac{1}{\binom{2n}{n}\sin^2(\theta_2-\theta_1)}\sum_{i=0}^{2n}a_i\phi_i^{2n}(\theta),
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
a_i=&\sum_{j=0}^{n}\binom{i}{j}\binom{2n-i}{n-j}c_j\left[b^{(2)}(i,j)c_{i-j+2}+b^{(1)}(i,j)c_{i-j+1}+b^{(0)}(i,j)c_{i-j}\right.\\
&\left.+b^{(-1)}(i,j)c_{i-j-1}+b^{(-2)}(i,j)c_{i-j-2}\right],
\end{aligned}
\tag{33}
$$

$c_k = 0$ for $k = -1, -2, \cdots, -n-2$ or $k = n+1, n+2, \cdots, 2n+2$, and $b^{(\ell)}_{i-j+\ell}$ can be found from equation (32) by using the combination formulas:

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} \quad or \quad \binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k}.$$

In fact, for $i < j$ or $j < i - n$, we have $b^{(\ell)}(i,j) = 0$; for $i - n \le j \le i$, we obtain

$$\begin{aligned}
b^{(2)}(i,j) =&(2n-i)(2n-i-1)\binom{2n-i-2}{n-j}\left(\binom{i+2}{j} - 3\binom{i}{j}\right) / \binom{i}{j}\binom{2n-i}{n-j} \\
=&(n-i+j)(n-i+j-1)\left(\frac{(i+2)(i+1)}{(i-j+2)(i-j+1)} - 3\right), \qquad (34)\\
b^{(1)}(i,j) =&2\cos(\theta_2-\theta_1)\left[i(2n-i)\binom{2n-i-1}{n-j}\left(\binom{i+1}{j} - 3\binom{i-1}{j}\right)\right. \\
&- (2n-i)(2n-i-1)\left(\binom{i+1}{j}\binom{2n-i-1}{n-j}\right. \\
&\left.\left.-3\binom{i}{j}\binom{2n-i-2}{n-j}\right)\right] / \binom{i}{j}\binom{2n-i}{n-j} \\
=&2(n-i+j)\cos(\theta_2-\theta_1)\left[\frac{(2i-2n+1)(i+1)}{i-j+1}\right. \\
&\left.+3(n-2i+2j-1)\right], \qquad (35)\\
b^{(0)}(i,j) =& \left[i(i-1)\binom{2n-i}{n-j}\left(\binom{i}{j} - 3\binom{i-2}{j}\right)\right. \\
&- 4i(2n-i)\cos^2(\theta_2-\theta_1)\left(\binom{i}{j}\binom{2n-i}{n-j} - 3\binom{i-1}{j}\binom{2n-i-1}{n-j}\right) \\
&+ (2n-i)(2n-i-1)\binom{i}{j}\left(\binom{2n-i}{n-j} - 3\binom{2n-i-2}{n-j}\right) \\
&\left.+(1-n^2)\sin^2(\theta_2-\theta_1)\binom{i}{j}\binom{2n-i}{n-j}\right] / \binom{i}{j}\binom{2n-i}{n-j} \\
=&i(i-1) - 3(i-j)(i-j-1) - 4i(2n-i)\cos^2(\theta_2-\theta_1) \\
&+ 12(i-j)(n-i+j)\cos^2(\theta_2-\theta_1) + (2n-i)(2n-i-1) \\
&- 3(n-i+j)(n-i+j-1) + (1-n^2)\sin^2(\theta_2-\theta_1) \\
=&i(i-1) + (2n-i)(2n-i-1) + 1 - n^2
\end{aligned}$$

$$- 3\left[(i-j)(i-j-1)+(n-i+j)(n-i+j-1)\right]$$
$$+ \cos^2(\theta_2-\theta_1)\left[n^2-1+4i(2n-i)-12(i-j)(n-i+j)\right], \tag{36}$$

$$\begin{aligned} b^{(-1)}(i,j) =& 2\cos(\theta_2-\theta_1)\left[-i(i-1)\left(\binom{i-1}{j}\binom{2n-i+1}{n-j}-3\binom{i-2}{j}\binom{2n-i}{n-j}\right)\right. \\ &\left. +i(2n-i)\binom{i-1}{j}\left(\binom{2n-i+1}{n-j}-3\binom{2n-i-1}{n-j}\right)\right]/\binom{i}{j}\binom{2n-i}{n-j} \\ =& 2(i-j)\cos(\theta_2-\theta_1)\left[\frac{(2n-2i+1)(2n-i+1)}{n-i+j+1}\right. \\ &\left. -3(n-2i+2j+1)\right], \end{aligned} \tag{37}$$

and

$$\begin{aligned} b^{(-2)}(i,j) =& i(i-1)\binom{i-2}{j}\left(\binom{2n-i+2}{n-j}-3\binom{2n-i}{n-j}\right)/\binom{i}{j}\binom{2n-i}{n-j} \\ =& (i-j)(i-j-1)\left[\frac{(2n-i+2)(2n-i+1)}{(n-i+j+2)(n-i+j+1)}-3\right]. \end{aligned} \tag{38}$$

From [7], we have the following positivity criterion for CBB polynomial $p(\theta)=\sum_{i=0}^{n} a_i\phi_i^n(\theta)$. If

$$a_0+(n-1)!\left(\frac{2}{n}\right)^{n-1}\sum_{\substack{i=1\\ a_i<0}}^{n-1}\frac{i^n}{i!(n-i)!}a_i\geq 0$$

and

$$a_n+(n-1)!\left(\frac{2}{n}\right)^{n-1}\sum_{\substack{i=1\\ a_i<0}}^{n-1}\frac{(n-i)^n}{i!(n-i)!}a_i\geq 0,$$

then $p(\theta)\geq 0$. Therefore, from Lemma 4 and equation (32) we obtain the following convexity criterion for $p(\theta)$.

Theorem 3. Let the $P(\theta)$ defined in (10) be a CBB curve and $p(\theta)$ be the associated CBB polynomial defined in equation (9). If

$$a_0+(2n-1)!\left(\frac{2}{2n}\right)^{2n-1}\sum_{\substack{i=1\\ a_i<0}}^{2n-1}\frac{i^{2n}}{i!(2n-i)!}a_i\geq 0$$

and

$$a_{2n} + (2n-1)!\left(\frac{2}{2n}\right)^{2n-1} \sum_{\substack{i=1 \\ a_i<0}}^{2n-1} \frac{(2n-i)^{2n}}{i!(2n-i)!} a_i \geq 0,$$

where a_i is defined by (33)-(38), then the CBB curve $P(\theta)$ is convex.

If we use the positivity criterion given in [11] (if

$$a_0 + (n-1)! \sum_{\substack{i=1 \\ a_i<0}}^{n-1} \frac{i}{i!(n-i)!} a_i \geq 0$$

and

$$a_n + (n-1)! \sum_{\substack{i=1 \\ a_i<0}}^{n-1} \frac{n-i}{i!(n-i)!} a_i \geq 0,$$

then $p(\theta) \geq 0$), we have another convexity criterion for $p(\theta)$, which is as follows.

Theorem 4. Let the $P(\theta)$ defined in (10) be a CBB curve and $p(\theta)$ be the associated CBB polynomial defined in equation (9). If

$$a_0 + (2n-1)! \sum_{\substack{i=1 \\ a_i<0}}^{2n-1} \frac{i}{i!(2n-i)!} a_i \geq 0$$

and

$$a_{2n} + (2n-1)! \sum_{\substack{i=1 \\ a_i<0}}^{2n-1} \frac{2n-i}{i!(2n-i)!} a_i \geq 0,$$

where a_i is defined by (33)-(38), then the CBB curve $P(\theta)$ is convex.

4. Convexity criteria of HBB polynomials

In Section 1, we have shown that if $\hat{T}$ is a trihedron generated by $\{v_1, v_2, v_3\}$ and if $b_1(v)$, $b_2(v)$, $b_3(v)$ denote the trihedron coordinates, i.e.,

$$\hat{T} = \{v \in R^3 : v = b_1 v_1 + b_2 v_2 + b_3 v_3,\ b_i \geq 0\},$$

then the HBB polynomials of degree n can be written as

$$p_n(v) = \sum_{|i|=n} a_i \phi_i^n(b). \tag{39}$$

Here ϕ_i^n was defined in equation (3) in Section 1. In this section, we will discuss the convexity criteria of HBB polynomial (39). Since a positively homogeneous convex function is called a gauge function, these convexity criteria can be also considered as conditions for making a HBB polynomial a gauge function (cf. [4]).

In the following, we will use the notation $D_\gamma = \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z}$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. For $\gamma = v_\ell$, $\ell = 1, 2, 3$, we denote $D_\ell = D_\gamma = D_{v_\ell}$. If we define that $E_\ell a_i = a_{i+\mathbf{e}^\ell}$, where $\mathbf{e}^\ell$ denotes the ℓ^{th} coordinate vector in $\mathbf{R}^3$, we have

$$D_\ell p_n = n \sum_{|i|=n-1} E_\ell a_i \phi_i^{n-1}(b). \tag{40}$$

For any direction V, there exists a vector $\mathbf{c}_V = (c_1, c_2, c_3)$ such that

$$V = \sum_{\ell=1}^{s} c_\ell v_\ell. \tag{41}$$

Thus, from (41), we have

$$D_V^2 p_n = n(n-1) \sum_{|i|=n-2} \mathbf{c}_V^T Q_{i,a} \mathbf{c}_V \phi_i^{n-2}(b), \tag{42}$$

where $c_V = (c_1, c_2, c_3)^T$ and

$$Q_{i,a} := (E_u E_w a_i)_{u,w=1}^{3,3} \tag{43}$$

for $|i| = n-2$.

Obviously, $p_n(v)$ is convex on $\hat{T}$ if and only if $D_V^2 p_n(v) \geq 0$ for any directional vector V and at any point $v \in \hat{T}$. Denoting $q_{i,a}(\mathbf{c}_V) = \mathbf{c}_V^T Q_{i,a} \mathbf{c}_V$, we have

$$D_V^2 p_n(v) = n(n-1) \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) \phi_i^{n-2}(b). \tag{44}$$

We define a function $\mathbf{c}_V$ associated with $q_{i,f}(\mathbf{c})$ as follows:

$$w_{i,a}(\mathbf{c}_V) = \begin{cases} 0, & \text{if } q_{i,a}(\mathbf{c}_V) \geq 0, \\ 1, & \text{if } q_{i,a}(\mathbf{c}_V) < 0, \end{cases} \tag{45}$$

where $|i| = n-2$ and $i_\ell \neq n-2$, $\ell = 1,2,3$. If $|i| = n-2$ and $i_\ell = n-2$ for $\ell = 1,2,3$, then $w_{i,a}(\mathbf{c}_V) = 1$.

The following two inequalities about $\phi_i^n(b)$, (46) and (47), were obtained in [7] and [11], respectively, by using inequalities from [6, p. 17].

$$0 \leq \phi_i^n(b) \leq \frac{(n-1)!}{i!n^{n-1}} (\sum_{\ell=1}^{3} i_\ell b_\ell)^n, \tag{46}$$

$$0 \leq \phi_i^n(b) \leq \frac{(n-1)!}{i!} \sum_{\ell=1}^{3} i_\ell b_\ell^n. \tag{47}$$

We now give some convexity criteria for the homogeneous Bernstein-Bézier polynomials over triangle $\hat{T}$.

Theorem 5. Let $r_i \in \{0,1\}$ for $|i| = n-2$ and $i \neq (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$, and $r_i = 1$ for $i = (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$. The Bernstein-Bézier polynomial $p_n(v)$ shown in (39) is convex on $\hat{T}$ if for all $u \in \{1,2,3\}$ its Bézier coefficients satisfy either

$$\sum_{|i|=n-2} \left(\sum_{\ell=0}^{s} i_\ell^2 \right)^{\frac{n-2}{2}} \frac{r_i}{i!} E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w \neq u}} \left| \sum_{|i|=n-2} \left(\sum_{\ell=0}^{s} i_\ell^2 \right)^{\frac{n-2}{2}} \frac{r_i}{i!} E_u E_w a_i \right| \tag{48}$$

or

$$\sum_{|i|=n-2} \left(\sum_{\ell=0}^{s} i_\ell^2 \right)^{\frac{n-2}{2}} \frac{r_i}{i!} \left(E_u E_u a_i - \sum_{\substack{w=1,2,3 \\ w \neq u}} |E_u E_w a_i| \right) \geq 0. \tag{49}$$

Proof. It is sufficient to prove inequality (48), since inequality (49) is implied by inequality (48). Noting inequality (46), we have

$$\begin{aligned}
&\frac{1}{n(n-1)} D_V^2 p_n(v) \\
&= \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V)\phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V)\phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V)\frac{(n-3)!}{i!(n-2)^{n-3}} \left(\sum_{\ell=1}^{3} i_\ell b_\ell\right)^{n-2} \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V)\frac{(n-3)!}{i!(n-2)^{n-3}} \left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \left(\sum_{\ell=1}^{3} b_\ell^2\right)^{\frac{n-2}{2}} \\
&= \frac{(n-3)!}{(n-2)^{n-3}} \left(\sum_{\ell=1}^{3} b_\ell^2\right)^{\frac{n-2}{2}} \mathbf{c}_V^T \left[\sum_{|i|=n-2} \left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} Q_{i,a}\right] \mathbf{c}_V \\
&= \frac{(n-3)!}{(n-2)^{n-3}} \left(\sum_{\ell=1}^{3} b_\ell^2\right)^{\frac{n-2}{2}} \mathbf{c}_V^T \left[\sum_{|i|=n-2} \left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} E_u E_w a_i\right]_{u,w=1}^{3,3} \mathbf{c}_V.
\end{aligned}$$

Obviously, if the last symmetric matrix is strongly diagonally dominant, i.e., for all $u = 1, 2, 3$,

$$\sum_{|i|=n-2} \left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w\neq u}} \left| \sum_{|i|=n-2} \left(\sum_{\ell=0}^{s} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V}{i!} E_u E_w a_i \right|,$$

then the matrix

$$\left[\sum_{|i|=n-2} \left(\sum_{\ell=1}^{3} i_\ell^2\right)^{\frac{n-2}{2}} \frac{w_{i,a}(\mathbf{c}_V)}{i!} E_u E_w a_i\right]_{u,w=1}^{3,3}$$

is semi-positive definite. Thus $D_V^2 p_n(v) \geq 0$ and $p_n(v)$ is convex. Obviously, the above condition is implied by inequality (48). Thus, Theorem 5 is proved.

Similarly, we may use inequality (30) to obtain the following result.

Theorem 6. Let $r_i \in \{0,1\}$ for $|i| = n-2$ and $i \neq (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$, and $r_i = 1$ for $i = (n-2)\mathbf{e}^\ell$, $\ell = 1,2,3$. The Bernstein-Bézier polynomial $p_n(v)$ shown in (35) is convex on $\hat{T}$ if for $\ell = 1,2,3$ and $u = 1,2,3$, its Bézier coefficients satisfy either

$$\sum_{|i|=n-2} \frac{i_\ell}{i!} r_i E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w\neq u}} \left| \sum_{|i|=n-2} \frac{i_\ell}{i!} r_i E_u E_w a_i \right| \tag{50}$$

or

$$\sum_{|i|=n-2} \frac{i_\ell}{i!} r_i \left(E_u E_u a_i - \sum_{\substack{w=1,2,3 \\ w\neq u}} |E_u E_w a_i| \right) \geq 0. \tag{51}$$

Proof. It is sufficient to prove inequality (50), since inequality (51) is implied by inequality (50). Noting inequality (47), we have

$$\begin{aligned}
&\frac{1}{n(n-1)} D_V^2 p_n(v) \\
&= \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) \phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V) \phi_i^{n-2}(b) \\
&\geq \sum_{|i|=n-2} q_{i,a}(\mathbf{c}_V) w_{i,a}(\mathbf{c}_V) \frac{(n-3)!}{i!} \sum_{\ell=1}^{3} i_\ell b_\ell^{n-2} \\
&= (n-3)! \sum_{\ell=1}^{3} b_\ell^{n-2} \mathbf{c}_V^T \left[\sum_{|i|=n-2} \frac{i_\ell}{i!} w_{i,a}(\mathbf{c}_V) Q_{i,a} \right] \mathbf{c}_V
\end{aligned}$$

$$= (n-3)! \sum_{\ell=1}^{3} b_\ell^{n-2} \mathbf{c}_V^T \left[\sum_{|i|=n-2} \frac{i_\ell}{i!} w_{i,a}(\mathbf{c}_V) E_u E_w a_i \right]_{u,w=1}^{3,3} \mathbf{c}_V$$

Obviously, if the last symmetric matrix is strongly diagonally dominant, i.e., for all $u = 1, 2, 3$,

$$\sum_{|i|=n-2} \frac{i_\ell}{i!} w_{i,a}(\mathbf{c}_V) E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w \neq u}} \left| \sum_{|i|=n-2} \frac{i!}{i!} w_{i,a}(\mathbf{c}_V) E_u E_w a_i \right|,$$

then the matrix

$$\left[\sum_{|i|=n-2} \frac{i!}{i!} w_{i,a}(\mathbf{c}_V) E_u E_w a_i \right]_{u,w=1}^{3,3}$$

is semi-positive definite. Thus $D_V^2 p_n(v) \geq 0$ and $p_n(v)$ is convex. Obviously, the above condition is implied by inequality (50). Thus, Theorem 6 is proved.

Remark 4. A stronger convexity condition is implied by inequalities (49) and (51) as follows: $E_u E_u a_i \geq \sum_{\substack{w=1,2,3 \\ w \neq u}} \mid E_u E_w a_i \mid$. Here, $u = 1, 2, 3$.

Remark 5. The conditions given in Theorem 5 and Theorem 6 are independent (see [7] and [8]).

Remark 6. There is another approach for finding convexity criteria from the positivity criteria and is shown in [8] for plane BB polynomials. This approach can also be applied here for HBB polynomials.

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REFERENCES

[1] P. Alfeld, M. Neamtu, and L. L. Schumaker, Bernstein-Bézier polynomial on spheres and sphere-like surfaces, *Comput. Aided Geom. Design*, 13 (1996), no. 4, 333–349.

[2] P. Alfeld, M. Neamtu, and L. L. Schumaker, Circular Bernstein-Bézier polynomials, *Mathematical Methods for Curves and Surfaces*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Vanderbilt University Press, 1995, 11-20.

[3] W. Dahmen, Convexity and Bernstein-Bézier polynomials, *Curves and Surfaces*, P. J. Laurent, A. Le Ḿhauté, and L. L. Schumaker (eds.), 1991, 107-134.

[4] H. G. Eggleston, *Convexity*, Cambridge Univ. Press, London, 1969.

[5] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design, A Practical Guide*, Academic Press, New York, 1988.

[6] G. H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1934.

[7] T. X. He, Shape criteria of Bernstein-Bézier polynomials over simplexes, *Comp. Math. Appl.*, (30)1995, 317-333.

[8] T. X. He, Positivity and convexity criteria for Bernstein-Bzier polynomials over simplices. Curves and surfaces with applications in CAGD (ChamonixMont-Blanc, 1996), 169176, Vanderbilt Univ. Press, Nashville, TN, 1997.

[9] C. C. Hsiung, *A First Course in Differential Geometry*, John Wiley & Sons, New York, 1981.

[10] S. R. Lay, *Convex Sets and Their Applications*, John Wiley & Sons, New York, 1982.

[11] Z. Wang and Q. Liu, An improved condition for the convexity and positivity of Bernstein-Bézier surfaces over triangles, *Comp. Aided Geom. Des.*, **5** (1988), 269-275.

[12] C. Z. Zhou, On the convexity of parametric Bézier triangular surfaces, *Comp. Aided Geom. Des.*, **7** (1990), 459-463.

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