

On an Extension of Abel-Gontscharoff's Expansion Formula

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Abstract

We present a constructive generalization of Abel-Gontscharoff's series expansion to higher dimensions. A constructive application to a problem of multivariate interpolation is also investigated. In addition, two algorithms for the constructing the basis functions of the interpolants are given.

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1 Introduction

Throughout we should adopt various notations in the multiple-index system.

(i) Given $\nu \in \mathbb{N}^s$ with $\nu = (\nu_1, \nu_2, \dots, \nu_s)$. We denote

$$|\nu| := \sum_{i=1}^s \nu_i, \quad \nu! := \nu_1! \nu_2! \cdots \nu_s!$$

If $k \in \mathbb{N}^s$ such that $k \leq \nu$ (i.e., $k_i \leq \nu_i$, $1 \leq i \leq s$), then we denote

$$\binom{\nu}{k} := \binom{\nu_1}{k_1} \binom{\nu_2}{k_2} \cdots \binom{\nu_s}{k_s} = \frac{\nu!}{k!(\nu - k)!},$$

where $\nu - k = (\nu_1 - k_1, \nu_2 - k_2, \dots, \nu_s - k_s)$.

(ii) Given $\nu \in \mathbb{N}^s$ and $x \in \mathbb{R}^s$ (or \mathbb{C}^s) with $x = (x_1, x_2, \dots, x_s)$. We denote

$$x^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots x_s^{\nu_s}.$$

(iii) Given $x, \alpha \in \mathbb{R}^s$ (or \mathbb{C}^s). We denote

$$\partial^\nu f(x) := \frac{\partial^{|\nu|} f(x)}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_s^{\nu_s}}, \quad \partial^\nu f(\alpha) = \partial^\nu f(x)|_{x=\alpha},$$

and if $\nu = \mathbf{0} = (0, \dots, 0)$ we denote $\partial^{\mathbf{0}} f(\alpha) = f(\alpha)$.

(iv) For $k \in \mathbb{N}^s$ we define $\alpha_k \in \mathbb{R}^s$ as a multiple sequence of dimension s , i.e.,

$$\alpha_k = (\alpha_k^{(1)}, \alpha_k^{(2)}, \dots, \alpha_k^{(s)}), \quad (k \in \mathbb{N}^s).$$

Also, we write $\infty = (\infty, \dots, \infty)$.

The multivariate Taylor-Maclaurin expansion at $\mathbf{0}$ is given by

$$f(x) = \sum_{\nu \geq \mathbf{0}} \frac{1}{\nu!} \partial^\nu f(\mathbf{0}) x^\nu \tag{1.1}$$

$$f(x) = \sum_{|\nu| \leq r} \frac{1}{\nu!} \partial^\nu f(\mathbf{0}) x^\nu + \sum_{|\nu|=r+1} \frac{1}{\nu!} \partial^\nu f(\theta x) x^\nu, \tag{1.2}$$

where $0 < \theta < 1$ and $\theta x = (\theta x_1, \theta x_2, \dots, \theta x_s)$, $r \geq 1$.

For the simple case $s = 1$, let $n, k \in \mathbb{N}$ and $x \in \mathbb{C}$. Given $\beta_k \in \mathbb{C}$. It is known that Gould's algebraic identity takes the form

$$\sum_{k=0}^n \binom{n}{k} c(k) (x - \beta_k)^{n-k} = x^n, \tag{1.3}$$

in which $c(0) = 1$, and $c(k) \equiv c(k; \beta) \equiv c(k; \beta_0, \beta_1, \dots, \beta_{k-1})$ is a kind of homogeneous polynomial, called the Abel-Gontscharoff-Gould polynomial, of degree k in $\beta_1, \beta_2, \dots, \beta_{k-1}$. For more details about $c(k)$, see Gould [2], Hsu [3], and He-Hsu-Shiue [4].

Let $\Gamma \equiv (\Gamma, +, \cdot)$ be the commutative ring of formal power series over \mathbb{C}^s (or \mathbb{R}^s), and let $\alpha_k \equiv (\alpha_{k_1}^{(1)}, \alpha_{k_2}^{(2)}, \dots, \alpha_{k_s}^{(s)})$ ($k \in \mathbb{N}^s$) be a given multiple sequence. Then, what a basic result to be proved in this paper (cf. Section 2) is the following: For any $f \in \Gamma$ we have a formal series expansion of the form

$$f(x) = \sum_{k \geq \mathbf{0}} \frac{\partial^k f(\alpha_k)}{k!} C(k; x - \alpha), \quad (1.4)$$

where $C(k; x - \alpha)$ is given by

$$\begin{aligned} & C(k; x - \alpha) \\ & := \prod_{i=1}^s c\left(k; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)}\right) \end{aligned} \quad (1.5)$$

Evidently, the classical Abel-Gontscharoff series expansion (i.e., Abel-Gontscharoff interpolation series) is a particular case of (1.4) and (1.5) with $s = 1$. Also we shall show that a multivariate polynomial $\Phi_r(x) \equiv \Phi_r(f; x) \in \pi_r^s$ (the set of all polynomials of degree $\leq k$ in s variables) of the form (with $r \geq 1$)

$$\Phi_r(f; x) = \sum_{|k| \leq r} \frac{\partial^k f(\alpha_k)}{k!} C(k; x - \alpha) \quad (1.6)$$

just solves a general problem for multivariate interpolation. This will be discussed latter in Section 2. Finally in Section 3, we give two algorithms for computing the Abel-Gontscharoff-Gould polynomials, i.e., the interpolation basis functions, defined by (1.5).

2 Two Main Theorems

The following lemmas are needed.

Lemma 2.1 (A multivariate form of Gould's identity) Let $x \in \mathbf{C}^s$ and $\nu, k \in \mathbf{N}^s$, and denote $\beta^{(i)} \equiv (\beta_0^{(i)}, \beta_1^{(i)}, \dots, \beta_{k_i-1}^{(i)})$, ($i = 1, 2, \dots$). We have an algebraic identity of the form

$$x^\nu = \sum_{k \leq \nu} \binom{\nu}{k} \prod_{i=1}^s c(k_i; \beta^{(i)}) (x_i - \beta_{k_i}^{(i)})^{\nu_i - k_i}. \quad (2.1)$$

Proof. Application of (1.3) to each of the s factors of $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_s^{\nu_s}$ yields the result expression (2.1). ■

Lemma 2.2 Let $\nu \in \mathbf{N}^s$, $x \in \mathbf{C}^s$, and $\alpha_k \equiv (\alpha_{k_1}^{(1)}, \alpha_{k_2}^{(2)}, \dots, \alpha_{k_s}^{(s)})$ with $k \in \mathbf{N}^s$. Then we have

$$\partial^\nu \prod_{i=1}^s c(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)}) \Big|_{x=\alpha_\nu} = \begin{cases} 0 & \text{if } \nu \neq k, \\ k! & \text{if } \nu = k. \end{cases} \quad (2.2)$$

Proof. This follows easily from a repeated application of the equation (2.2) of our paper [4]. Indeed, in accordance with Proposition 2.1 of [4] we see that the left-hand side of (2.2) can be re-written in the form

$$\begin{aligned} & \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_s^{\nu_s}} \prod_{i=1}^s Q_{k_i}(x_i) \Big|_{x=\alpha_\nu} = \prod_{i=1}^s Q_{k_i}^{(\nu_i)}(\alpha_{\nu_i}^{(i)}) \\ & = \prod_{i=1}^s \nu_i! \delta_{\nu_i k_i} = \begin{cases} 0 & \text{if } \nu \neq k, \\ k! & \text{if } \nu = k, \end{cases} \end{aligned}$$

wherein $\delta_{\cdot, \cdot}$ is the Kronecker symbol. ■

Theorem 2.3 Let $\{\alpha_k\}$ be a given s -multiple sequence with $\alpha_k \equiv (\alpha_{k_1}^{(1)}, \alpha_{k_2}^{(2)}, \dots, \alpha_{k_s}^{(s)}) \in \mathbf{C}^s$ (or \mathbb{R}^s) and $k \in \mathbf{N}^s$. Then for any $f \in \Gamma(\mathbf{C}^s)$ we have the formal series expansion formula

$$f(x) = \sum_{k \geq \mathbf{0}} \frac{\partial^k f(\alpha_k)}{k!} \Pi_{i=1}^s c \left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)} \right), \quad (2.3)$$

where $x \in \mathbb{C}^s$ (or \mathbb{R}^s), and $c \left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)} \right)$ are Abel-Gontscharoff-Gould polynomials with degrees $k_i \in \mathbb{N}$.

Proof. For brevity let us denote

$$\beta^{(i)} \equiv \left(\beta_0^{(i)}, \beta_1^{(i)}, \dots, \beta_{k_i-1}^{(i)} \right), \quad C(k; \beta) \equiv \Pi_{i=1}^s c(k_i; \beta^{(i)}).$$

Also we shall make the substitutions $\beta_j^{(i)} = x_i - \alpha_j^{(i)}$, ($i, j \in \mathbb{N}$). Then using Lemma 1.1 and the multivariate Taylor expansion, we see that $f(x)$ can be formally expanded as follows.

$$\begin{aligned} f(x) &= \sum_{\nu \geq \mathbf{0}} \frac{\partial^\nu f(\mathbf{0})}{\nu!} x^\nu \\ &= \sum_{\nu \geq \mathbf{0}} \frac{\partial^\nu f(\mathbf{0})}{\nu!} \sum_{k \leq \nu} \binom{\nu}{k} \Pi_{i=1}^s c(k_i; \beta^{(i)}) \left(x_i - \beta_{k_i}^{(i)} \right)^{\nu_i - k_i} \\ &= \sum_{k \geq \mathbf{0}} \frac{C(k; \beta)}{k!} \sum_{\nu \geq k} \frac{\partial^\nu f(\mathbf{0})}{(\nu - k)!} \left(x_i - \beta_{k_i}^{(i)} \right)^{\nu_i - k_i} \end{aligned}$$

By substituting $\mu = \nu - k = (\mu_1, \mu_2, \dots, \mu_s)$ into the right-hand side of the last equation we can write it as

$$\begin{aligned} & \sum_{k \geq \mathbf{0}} \frac{C(k; \beta)}{k!} \sum_{\mu \geq \mathbf{0}} \frac{\partial^{\mu+k} f(\mathbf{0})}{\mu!} \left(x_i - \beta_{k_i}^{(i)} \right)^{\mu_i} \\ &= \sum_{k \geq \mathbf{0}} \frac{C(k; \beta)}{k!} \sum_{\mu \geq \mathbf{0}} \frac{1}{\mu!} \partial^\mu (\partial^k f(\mathbf{0})) \left(x_i - \beta_{k_i}^{(i)} \right)^{\mu_i} \\ &= \sum_{k \geq \mathbf{0}} \frac{C(k; \beta)}{k!} \partial^k f \left(x_1 - \beta_{k_1}^{(1)}, x_2 - \beta_{k_2}^{(2)}, \dots, x_s - \beta_{k_s}^{(s)} \right) \\ &= \sum_{k \geq \mathbf{0}} \frac{1}{k!} \partial^k f \left(\alpha_{k_1}^{(1)}, \alpha_{k_2}^{(2)}, \dots, \alpha_{k_s}^{(s)} \right) \\ & \quad \times \Pi_{i=1}^s c \left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)} \right), \end{aligned}$$

which is just the right-hand side of Eq. (2.3). ■

For the case $\alpha_k \equiv (a_1, a_2, \dots, a_s) \equiv a \in \mathbb{C}^s$, so that $\alpha_j^{(i)} = a_i$, we see that (2.3) implies the multivariate Taylor expansion as a consequence, namely

$$f(x) = \sum_{k \geq \mathbf{0}} \frac{\partial^k f(kt)}{k!} \prod_{i=1}^s x_i (x_i - k_i t_i)^{k_i-1}, \quad (2.4)$$

where $k \in \mathbb{N}^s$, $x, t \in \mathbb{C}^s$ and $kt \equiv (k_1 t_1, k_2 t_2, \dots, k_s t_s)$.

Theorem 2.4 *Given $\alpha_k \equiv (\alpha_{k_1}^{(1)}, \alpha_{k_2}^{(2)}, \dots, \alpha_{k_s}^{(s)}) \in \mathbb{C}^s$ with $k \in \mathbb{N}^s$, let $f(x) \in \Gamma$ (over \mathbb{C}^s). Then the s -variate Abel-Gontscharoff polynomial of degree r ($r \geq 1$) given by*

$$\begin{aligned} \Phi_r(x) &= \Phi_r(f; x) \\ &= \sum_{|k| \leq r} \frac{\partial^k f(\alpha_k)}{k!} \prod_{i=1}^s c\left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)}\right) \end{aligned} \quad (2.5)$$

satisfies the interpolation conditions

$$\begin{cases} \Phi_r(\alpha_0) = f(\alpha_0), & \left(\alpha_0 \equiv (\alpha_0^{(1)}, \alpha_0^{(2)}, \dots, \alpha_0^{(s)})\right), \\ \partial^\nu \Phi_r(x)|_{x=\alpha_\nu} = \partial^\nu f(\alpha_\nu), & 1 \leq |\nu| \leq r. \end{cases} \quad (2.6)$$

Proof. In the first place, notice that $c(j; \beta_0, \beta_1, \dots, \beta_{j-1}) = 0$ for $j \in \mathbb{N}$, $j \geq 1$ and $\beta_0 = 0$. Thus we have

$$\prod_{i=1}^s c\left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)}\right) \Big|_{x=\alpha_0} = 0, \quad (|k| \geq 1).$$

Moreover, $c(0; \beta) = 1$. So it follows that

$$\Phi_r(\alpha_0) = \Phi_r(f; x)|_{x=\alpha_0} = \partial^0 f(\alpha_0) = f(\alpha_0).$$

Furthermore, for $|\nu| \geq 1$ we have by using Lemma 2.2

$$\begin{aligned}
 & \partial^\nu \Phi_r(x) \Big|_{x=\alpha_\nu} \\
 = & \sum_{|k| \leq r} \frac{\partial^k f(\alpha_k)}{k!} \partial^\nu \Pi_{i=1}^s c \left(k_i; x_i - \alpha_0^{(i)}, x_i - \alpha_1^{(i)}, \dots, x_i - \alpha_{k_i-1}^{(i)} \right) \Big|_{x=\alpha_\nu} \\
 = & \frac{\partial^k f(\alpha_k)}{k!} \left\{ \begin{array}{ll} 0 & (k \neq \nu) \\ k! & (k = \nu) \end{array} \right\} = \partial^\nu f(\alpha_\nu).
 \end{aligned}$$

Hence Theorem 2.4 is proved. ■

With the notation given by (1.5) the polynomial defined by (2.5) may be written in a more compact form, namely

$$\Phi_r(f; x) = \sum_{|k| \leq r} \frac{\partial^k f(\alpha_k)}{k!} C(k; x - \alpha_0, x - \alpha_1, \dots, x - \alpha_{k-1}), \quad (2.7)$$

where $x \in \mathbb{C}^s$, $k \in \mathbb{N}^s$, and $\{\alpha_k\}$ is a given sequence with $\alpha_k \in \mathbb{C}^s$. Certainly, $\Phi_r(f; x)$ may be called the s -variate Abel-Gontscharoff interpolation polynomial of degree r .

Remark 2.1 The difference $\rho_r(f; x) = f(x) - \Phi_r(f; x)$ is called a remainder of the expression (2.3). From the viewpoint of numerical analysis, it may be of interest to find some useful expression for $\rho_r(f; x)$ ($r \geq 1$). In particular, as an unsolved problem, we propose to investigate whether there is a remainder formula of Lagrange form involving that of (1.2) as a particular case. However, as a special case for that all α_k are the same when $|k|$ fixed, the corresponding remainder will be shown in the next section.

We now extend the dot product annihilation coefficients, defined by Gould in [2], to the higher dimension setting. Similar to [2], we define the multivariate dot product annihilation coefficients $\alpha(\nu, k; \beta) \equiv \alpha(\nu, k; \beta_0, \dots, \beta_\nu)$ ($\nu, k \in \mathbb{N}^s$, $\beta_0, \dots, \beta_\nu \in \mathbb{C}^s$) by the expansion

$$\sum_{k \leq \nu} \alpha(\nu, k; \beta) (x - \beta_k)^k = x^\nu.$$

The inverse expansion is given by

$$\sum_{k \leq \nu} \binom{\nu}{k} (-\beta_\nu)^{\nu-k} x^k = (x - \beta_\nu)^\nu.$$

In terms of inverse relations we have the reciprocal pair

$$f_\nu = \sum_{k \leq \nu} \alpha(\nu, k; \beta) g_k \iff g_\nu = \sum_{k \leq \nu} \binom{\nu}{k} (-\beta_\nu)^{\nu-k} f_k. \quad (2.8)$$

Evidently, the binomial inversion is just a simple particular case when $\beta_\nu = \text{constant vector} \neq \mathbf{0}$, ($\nu \in \mathbb{N}^s$).

An identity can be derived from the reciprocal pair shown as in (2.8). Substituting the second equation in (2.8) to the first one yields

$$\begin{aligned} f_\nu &= \sum_{k \leq \nu} \alpha(\nu, k; \beta) \sum_{\ell \leq k} (-\beta_k)^{k-\ell} \binom{k}{\ell} f_\ell \\ &= \sum_{\ell \leq \nu} \left[\sum_{\ell \leq k \leq \nu} \alpha(\nu, k; \beta) (-\beta_k)^{k-\ell} \binom{k}{\ell} \right] f_\ell \end{aligned}$$

Thus, we obtain the following identity

$$\sum_{\ell \leq k \leq \nu} \alpha(\nu, k; \beta) (-\beta_k)^{k-\ell} \binom{k}{\ell} = \delta_{\nu\ell}. \quad (2.9)$$

where $\nu, \ell \in \mathbb{N}^s \cup \{\mathbf{0}\}$, and $\delta_{\nu\ell} = \prod_{i=1}^s \delta_{\nu_i \ell_i}$, and $\delta_{\nu_i \ell_i}$ ($i = 1, \dots, s$) are the Kronecker symbol; i.e., $\delta_{\nu_i \ell_i}$ equals to 1 if $\nu_i = \ell_i$ and 0 otherwise.

3 Computation of the Abel-Gontscharoff-Gould polynomials

In order to make (2.7) (or (2.3)) really available, it needs to devise some algorithms for computing the Abel-Gontscharoff-Gould polynomials defined by (1.5) and (1.3). In what follows we shall discuss computational aspects of Abel-Gontscharoff-Gould polynomials. Two algorithms for computing the Abel-Gontscharoff-Gould polynomials (hence, the basis functions of the multivariate Abel-Gontscharoff interpolation shown in Theorem 2.4) will be given.

By checking the first few expressions of $c(n)$ we find the relation shown in the following Theorem 3.1, which can be used to establish a general expression of $c(n)$ recursively and write them out readily.

Theorem 3.1 *For all $n \in \mathbb{N}$ and $\beta_k \in \mathbb{C}$ ($0 \leq k \leq n$). The Abel-Gontscharoff-Gould polynomials shown in (1.3) can be written as*

$$c(n) = \sum_{j=1}^n \binom{n}{j} (-1)^{j-1} \beta_0^j P^j c(n-j), \quad (3.1)$$

where the index operator P is defined as

$$\begin{aligned} Pc(n) &\equiv P \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} c(j) \beta_j^{n-j} \\ &:= \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} Pc(j) \beta_{j+1}^{n-j}, \end{aligned} \quad (3.2)$$

and $P^{j+1}c(n) := P[P^j c(n)]$.

From Eq. (3.2) we have $Pc(0) = 1$ and

$$Pc(1) \equiv P \sum_{j=0}^0 (-1)^{-j} \binom{1}{j} c(j) \beta_j^{1-j} = P(c(0)\beta_0) := Pc(0)\beta_1 = \beta_1.$$

Hence, from (3.1), we can write the expression of $c(n)$ ($n = 1, 2, 3, 4$) recursively as

$$\begin{aligned} c(1) &= \beta_0 Pc(0) = \beta_0, \\ c(2) &= 2\beta_0 Pc(1) + (-1)\beta_0^2 Pc(0) = 2\beta_0\beta_1 - \beta_0^2, \\ c(3) &= 3\beta_0 Pc(2) - 3\beta_0^2 P^2 c(1) + \beta_0^3 P^3 c(0) \\ &= 3\beta_0\beta_1(2\beta_2 - \beta_1) - 3\beta_2\beta_0^2 + \beta_0^3, \\ c(4) &= 4\beta_0 Pc(3) - 6\beta_0^2 P^2 c(2) + 4\beta_0^3 P^3 c(1) - \beta_0^4 P^4 c(0) \\ &= 4\beta_0\beta_1 [3\beta_2(2\beta_3 - \beta_2) - 3\beta_1\beta_3 + \beta_1^2] - 6\beta_0^2\beta_2(2\beta_3 - \beta_2) + 4\beta_0^3\beta_3 - \beta_0^4, \\ c(5) &= 5\beta_0 Pc(4) - 10\beta_0^2 P^2 c(3) + 10\beta_0^3 P^3 c(2) - 5\beta_0^4 P^4 c(1) + \beta_0^5 P^5 c(0) \\ &= 5\beta_0 [4\beta_1\beta_2 (3\beta_3(2\beta_4 - \beta_3) - 3\beta_2\beta_4 + \beta_2^2) - 6\beta_1^2\beta_3(2\beta_4 - \beta_3) \\ &\quad + 4\beta_1^3\beta_4 - \beta_1^4] - 10\beta_0^2 [3\beta_2\beta_3(2\beta_4 - \beta_3) - 3\beta_4\beta_2^2 + \beta_2^3] \\ &\quad + 10\beta_0^3 [\beta_3(2\beta_4 - \beta_3)] - 5\beta_4\beta_0^4 + \beta_0^5. \end{aligned}$$

Proof. We now prove formula (3.1) by using the mathematical induction. It is easy to see the formula holds for $n = 1$ from the definition (3.2) of the operator P . Assume that the formula is true for all $1 \leq j \leq m$. We shall show it is also true for all $1 \leq j \leq m + 1$. In fact, using the induction assumption yields

$$\begin{aligned}
& c(m+1) \\
&= \sum_{j=0}^m (-1)^{m-j} \binom{m+1}{j} c(j) \beta_j^{m+1-j} \\
&= \sum_{j=1}^m (-1)^{m-j} \binom{m+1}{j} c(j) \beta_j^{m+1-j} + (-1)^m \beta_0^{m+1} \\
&= \sum_{j=1}^m (-1)^{m-j} \binom{m+1}{j} \beta_j^{m+1-j} \sum_{u=1}^j (-1)^{u-1} \binom{j}{u} \beta_0^u P^u c(j-u) + (-1)^m \beta_0^{m+1} \\
&= \sum_{u=1}^m \left(\sum_{j=u}^m (-1)^{m+u-j-1} \binom{m+1}{j} \binom{j}{u} \beta_j^{m+1-j} P^u c(j-u) \right) \beta_0^u + (-1)^m \beta_0^{m+1}.
\end{aligned}$$

For $1 \leq u \leq n$, the sum in the parentheses of the rightmost equality can be simplified as

$$\begin{aligned}
& \sum_{j=0}^{m-u} (-1)^{m-j-1} j \binom{m+1}{j+u} \binom{j+u}{u} \beta_{j+u}^{m-u+1-j} P^u c(j) \\
&= (-1)^{u-1} \binom{m+1}{u} \sum_{j=0}^{m-u} (-1)^{m-u-j} \binom{m-u+1}{j} \beta_{j+u}^{m-u+1-j} P^u c(j) \\
&= (-1)^{u-1} \binom{m+1}{u} P^u c(m-u+1).
\end{aligned}$$

Hence,

$$c(m+1) = \sum_{u=1}^m (-1)^{u-1} \binom{m+1}{u} \beta_0^u P^u c(m-u+1) + (-1)^m \beta_0^{m+1},$$

which completes the proof of the theorem.

■

Another algorithm is based on an alternative form of Proposition 2.2 and Remark 3 in [4], in which two expressions of $c(n)$ are given by using a determinant and the Hadamard product form of the determinant.

To give our second algorithm, we start from the Gould's algebraic identity (1.3) with the form

$$\sum_{k=0}^n \binom{n}{k} c(k) (x - \beta_k)^{n-k} = x^n, \quad (3.3)$$

where $n \in \mathbb{N}$ and $x, \beta_k \in \mathbb{C}$ ($0 \leq k \leq n$). For any fixed $n \in \mathbb{N}$, taking $x = 0, 1, \dots, n$ in (3.3) yields system

$$AC = G,$$

where $C, G \in \mathbb{R}^{n+1}$, $C = (c(n), c(n-1), \dots, c(1), c(0))^T$, $G = (0, 1, 2^n, \dots, n^n)^T$, and the column vectors of matrix A are of the form

$$(f_{n,k}(0), f_{n,k}(1), \dots, f_{n,k}(n))^T \quad (0 \leq k \leq n)$$

with $f_{n,k}(\ell) = \binom{n}{k} (\ell - \beta_k)^{n-k}$. In the following we denote the difference operator by Δ , i.e., $\Delta f(t) = f(t+1) - f(t)$, $\Delta^k f(t) = \Delta^{k-1} \Delta f(t)$. In general, $\Delta^k f(t)$ can be expressed as

$$\Delta^k f(t) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(t+j).$$

We also introduce the vectors

$$f_{n,k}^{[n]} := (f_{n,k}(0), \Delta f_{n,k}(0), \Delta^2 f_{n,k}(0), \dots, \Delta^n f_{n,k}(0))^T \quad (0 \leq k \leq n) \quad (3.4)$$

and the Vandermonde determinant

$$\tau(f_n) := \det \left[f_{n,n}^{[n]}, f_{n,n-1}^{[n]}, \dots, f_{n,1}^{[n]}, f_{n,0}^{[n]} \right]. \quad (3.5)$$

It can be seen that

$$\tau(f_n) = \prod_{k=0}^n \frac{n!}{k!} = \frac{n!}{\prod_{k=0}^n k!}$$

because the matrix shown in (3.5) is an upper triangle matrix with diagonal entries $\Delta^u f_{n,n-u}(0) = n!/(n-u)!$. Likewise, we denote $g_n(\ell) := \ell^n$,

$$g_n^{[n]} := (g_n(0), \Delta g_n(0), \dots, \Delta^n g_n(0))^T, \quad (3.6)$$

and

$$\tau_j(f_n|g_n) := \det \begin{bmatrix} f_{n,n}^{[n]}, \dots, f_{n,j+1}^{[n]}, g_n^{[n]}, f_{n,j-1}^{[n]}, \dots, f_{n,0}^{[n]} \end{bmatrix} \quad (3.7)$$

for $0 \leq j \leq n$. Hence, we have

Proposition 3.2 *Let $n \in \mathbb{N}$ and $x, \beta_k \in \mathbb{C}$ ($0 \leq k \leq n$). By using the notations $\tau(f_n)$ and $\tau_j(f_n|g_n)$ ($0 \leq j \leq n$) shown in (3.5) and (3.7), respectively, we have the following expressions of the Abel-Gontscharoff-Gould polynomials defined by (3.3).*

$$c(j) = \frac{\tau_{n-j}(f_n|g_n) \prod_{k=0}^n k!}{(n!)^n}. \quad (3.8)$$

Proof. Obviously, $\det A = \tau(f_n)$. Hence, for $0 \leq j \leq n$

$$c(j) = \frac{\tau_{n-j}(f_n|g_n)}{\det A} = \frac{\tau_{n-j}(f_n|g_n)}{\tau(f_n)},$$

and the Eq. (3.8) immediately follows. ■

As examples, from (3.8) we have

$$c(0) = \frac{\tau_n(f_n|g_n) \prod_{k=0}^n k!}{(n!)^n} = 1$$

and

$$\begin{aligned} c(1) &= \frac{\frac{(n!)^{n-1}}{\prod_{k=2}^n k!} (\Delta^n f_{n,0}(0) \Delta^{n-1} g_n(0) - \Delta^{n-1} f_{n,0}(0) \Delta^n g_n(0))}{\frac{(n!)^n}{\prod_{k=0}^n k!}}}{n!} \\ &= \frac{\Delta^{n-1} g_n(0) - \Delta^{n-1} f_{n,0}(0)}{n!} = \beta_0. \end{aligned}$$

Similarly,

$$\begin{aligned} c(2) &= \beta_0(2\beta_1 - \beta_0) \\ c(3) &= 3\beta_0\beta_1(2\beta_2 - \beta_1) - 3\beta_2\beta_0^2 + \beta_0^3 \\ c(4) &= 4\beta_0\beta_1 [3\beta_2(2\beta_3 - \beta_2) - 3\beta_1\beta_3 + \beta_1^2] \\ &\quad - 6\beta_0^2\beta_2(2\beta_3 - \beta_2) + 4\beta_0^3\beta_3 - \beta_0^4. \end{aligned}$$

Remark 3.1 There are still two other types of multivariate generalization of Abel-Gontscharoff interpolation series. One of them has been mentioned briefly in Remark 8 of our paper [4]. The second one has been given in a recent paper by one of the authors in [5], in which the classic Abel-Gontscharoff interpolation is extended to the multivariate Kergin interpolation by using a differential operator generated from Abel-Gontscharoff-Gould polynomial. Kergin type multivariate Abel-Gontscharoff interpolation was first studied by Cavaretta, Micchelli and Sharma in [1], in which a method for extending univariate interpolation procedures to higher dimensions was given. The idea is based on the requirement that the multivariate extension is related to its univariate analog on the class of ridge functions. In particular, the implicit multivariate Abel-Gontscharoff without the remainder was established in [1]. We shall show in another paper the relation between the multivariate Abel-Gontscharoff interpolation and the Kergin type Abel-Gontscharoff interpolation as well as the convergence rate of the multivariate Abel-Gontscharoff interpolation.

References

- [1] Cavaretta, Jr., A. S., Micchelli, C. A. and Sharma, A., Multivariate interpolation and the radon transform, Part II: Some Further Examples, Quantitative Approximation, R. A. Devore and K. Scherer (eds.), Academic Press, New York, 1980, 49-62.
- [2] Gould, H. W., Annihilation coefficients, *Analysis, Combinatorics and Computing*, (T. X. He, P. J. S. Shiue and Z. Li (eds)), Nova Sci. Publ. Inc., New York, 2002, 205-223.
- [3] Hsu, L. C., A general expansion formula, *Analysis, Combinatorics and Computing*, (T. X. He, P. J. S. Shiue and Z. Li (eds)), Nova Sci. Publ. Inc., New York, 2002, 251-258.

- [4] He, T.-X., Hsu, L.C., and Shiue, P. J.-S., On Abel-Gontscharoff-Gould's polynomials, *Analysis in Theory and Applications*, **19:2**(2003), 166-184.
- [5] He, T.-X., On multivariate Abel-Gontscharoff interpolation, *Advances in Constructive Approximation*, manuscript, 2003.