Construction of Nonlinear Expression for Recursive Number Sequences

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Dedicated to Professor Leetsch C. Hsu for the occasion of his 95th birthday.

Abstract

A type of nonlinear expressions of Lucas sequences are established inspired by Hsu [9]. Using the relationships between the Lucas sequence and other linear recurring sequences satisfying the same recurrence relation of order 2, we may transfer the identities of Lucas sequences to the latter.

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1 Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called sequence of order 2 if it satisfies the linear recurrence relation of order 2

$$a_n = p_1 a_{n-1} + p_2 a_{n-2}, \quad n \geq 2, \tag{1}$$

for some constants p_j $(j = 1, 2, \ldots, r)$, $p_2 \neq 0$, with initial vector (a_0, a_1). Linear recurrence relations with constant coefficients are important in subjects including combinatorics, pseudo-random number generation, circuit design, and cryptography, and they have been studied extensively. To construct an explicit formula of the general term of a number sequence of order r, one may use generating functions, characteristic equations, or matrix method (See Comtet [4], Hsu [8], Niven, Zuckerman, and Montgomery [11], Strang [12], Wilf [13], etc.) Recently, Shiue and the author give a reduction order method in [6]. Let A_2 be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_2 = \{p_1, p_2\}$. To study the structure of A_2 with respect to E_2, we make use of the Lucas sequence $\tilde{F}_n$ and its conjugate $\tilde{L}_n$ in A_2, which are particular sequences in A_2 with initials $a_0 = 0$ and $a_1 = 1$ and the initials $a_0 = 2$ and $a_1 = p_1$.

In next section, we will give the generating function and the expression of the Lucas sequences $\tilde{F}_n$ and $\tilde{L}_n$ and find out the relationships between them and the sequences in the set A_2 with the same E_2. In Section 3, with using the symbolic method shown in [9], we derive a type of identities of Lucas sequences in A_2 including a type of nonlinear expressions. The relationship between the Lucas sequences and other linear recurring sequences in the same set is used to transfer the identities of Lucas sequences to those of the linear recurring sequences in the same set.

2 Impulse response sequences

Among all the homogeneous linear recurring sequences satisfying second order homogeneous linear recurrence relation (1) with a nonzero p_1 and arbitrary initials $\{a_0, a_1\}$, the Lucas sequence with respect to $E_2 = \{p_1, p_2\}$ is the sequence satisfying (1) with initials $a_0 = 0$ and $a_1 = 1$ or the initial vector $(a_0, a_1) = (0, 1)$. For instance, Fibonacci sequence $\{F_n\}_{n\geq 0}$ is the Lucas sequence with respect to $\{1, 1\}$, Pell number sequence $\{P_n\}_{n\geq 0}$ is the Lucas sequence with respect to $\{2, 1\}$, and Jacobathal number sequence $\{J_n\}_{n\geq 0}$ is the Lucas sequence with respect to $\{1, 2\}$. For this reason, we may consider an Lucas sequence with respect to E_2 as an extension of Fibonacci number sequence and denoted it by $\{\tilde{F}_n\}_{n\geq 0}$, namely, $\tilde{F}_n$ satisfies (1) with initials $\tilde{F}_0 = 0$ and $\tilde{F}_1 = 1$.

In the following, we will present the structure of the linear recurring sequences defined by (1) using their characteristic polynomial. Then, we may find the relationship of those sequences with their corresponding Lucas sequences.

Proposition 2.1 *Let $\{a_n\} \in A_2$, i.e., let $\{a_n\}$ be the linear recurring sequence defined by (1). Then its generating function $P_2(t)$ can be written as*

$$P_r(t) = \frac{a_0 + (a_1 - p_1 a_0)t}{1 - p_1 t - p_2 t^2}. \tag{2}$$

Hence, the generating function for the Lucas sequence with respect to $\{p_1, p_2\}$ is

$$\tilde{P}_r(t) = \frac{t^{r-1}}{1 - p_1 t - p_2 t^2}. \tag{3}$$

Proof. (2) is easily to be checked by multiplying $1 - p_1 t - p_2 t^2$ on its both sides and noting

$$(1 - p_1 t - p_2 t^2) \sum_{n\geq 0} a_n t^n = \sum_{n\geq 0} a_n t^n - \sum_{n\geq 1} p_1 a_{n-1} t^n - \sum_{n\geq 2} a_{n-2} t^n$$
$$= a_0 + a_1 t - p_1 a_0 t + \sum_{n\geq 2} (a_n - p_1 a_{n-1} - p_2 a_{n-2}) t^n = a_0 + (a_1 - p_1 a_0)t.$$

By substituting $a_0 = 0$ and $a_1 = 1$ into (2), we obtain (3).

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We now give the explicit expression of $\tilde{F}_n$ in terms of the roots of the characteristic polynomial of recurrence relation shown in (1) as well as the relationships between the Lucas sequence and the recurring sequences in the set A_2 with the same E_2.

Proposition 2.2 *Let A_2 be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_2 = \{p_1, p_2\}$, and let $\{\tilde{F}_n\}$ be the Lucas sequence of A_2. Suppose α and β are two roots of the characteristic polynomial of A_2, which do not need to be distinct. Then*

$$\tilde{F}_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta}, & if\ \alpha \neq \beta; \\ n\alpha^{n-1}, & if\ \alpha = \beta. \end{cases} \tag{4}$$

In addition, every $\{a_n\} \in A_2$ can be written as

$$a_n = a_1 \tilde{F}_n - \alpha\beta a_0 \tilde{F}_{n-1}, \tag{5}$$

and a_n reduces to $a_1 \tilde{F}_n - \alpha^2 a_0 \tilde{F}_{n-1}$ when $\alpha = \beta$.

Conversely, there holds a expression of $\tilde{F}_n$ *in terms of* $\{a_n\}$ *as*

$$\tilde{F}_n = c_1 a_{n+1} + c_2 a_{n-1}, \tag{6}$$

where

$$c_1 = \frac{a_1 - a_0 p_1}{p_1(a_1^2 - a_0 a_1 p_1 - a_0^2 p_2)}, \quad c_2 = -\frac{a_1 p_2}{p_1(a_1^2 - a_0 a_1 p_1 - a_0^2 p_2)}, \tag{7}$$

provided that $p_1 \neq 0$*, and* $a_1^2 - a_0 a_1 p_1 - a_0^2 p_2 \neq 0$*.*

Proof. Recall that [6] presented the following result in its Proposition 2.1:

$$a_n = \begin{cases} \left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & if\ \alpha \neq \beta; \\ n a_1 \alpha^{n-1} - (n-1) a_0 \alpha^n, & if\ \alpha = \beta, \end{cases} \tag{8}$$

for every $\{a_n\} \subset A_2$. By substituting $a_0 = 0$ and $a_1 = 1$ into (8), one may obtain (4).

Denote by $L : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$ the operator $L(a_{n-1}, a_{n-2}) := p_1 a_{n-1} + p_2 a_{n-2} = a_n$. It is obvious that L is linear, and the sequence $\{a_n\}$ is uniquely determined by L from a given initial vector $(a_0,\, a_1)$. Define $a_{-1} = (a_1 - p_1 a_0)/p_2$, then (a_{-1}, a_0) is the initial vector that generates $\{a_{n-1}\}_{n\geq 0}$ by L. Similarly, the vector $(a_1, p_1 a_1 + p_2 a_0)$ generates sequence $\{a_{n+1}\}_{n\geq}$ by using L. Note the initial vectors of $\tilde{F}_n$ is $(0, 1)$. Thus (6) holds if and only if the initial vectors on the two sides are equal:

$$(0, 1) = c_1(a_1, p_1 a_1 + p_2 a_0) + c_2 \left(\frac{a_1 - p_1 a_0}{p_2}, a_0\right), \tag{9}$$

which yields the solutions (7) for c_1 and c_2 and completes the proof of the corollary.

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Proposition 2.2 presents the interrelationship between a linear recurring sequence with respect to $E_2 = \{p_1, p_2)$ and its Lucas sequence, which can be used to establish the identities of one sequence from the identities of other sequences in the same set.

Example 2.1 Let us consider A_2, the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_2 = \{p_1, p_2\}$. If $E_2 = \{1, 1\}$, then the corresponding characteristic polynomial has roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, and (6) gives the expression of the ISR of A_2, which is Fibonacci sequence $\{F_n\}$:

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

The sequence in A_2 with the initial vector $(2,1)$ is Lucas sequence $\{L_n\}$. From (5) and (6) and noting $\alpha\beta = -1$, we have the well-known formulas (see, for example, [10]):

$$L_n = F_n + 2F_{n-1} = F_{n+1} + F_{n-1}, \quad F_n = \frac{1}{5}L_{n+1} + \frac{1}{5}L_{n-1}. \tag{10}$$

By using the above formulas, one may transfer identities of Fibonacci number sequence to those of Lucas number sequence and vice verse. For instance, the above relationship can be used to prove that the following two identities are equivalent:

$$\begin{aligned} &F_{n+1}F_{n+2} - F_{n-1}F_n = F_{2n+1} \\ &L_{n+1}^2 + L_n^2 = L_{2n} + L_{2n+2}. \end{aligned}$$

It is clear that both of the identities are equivalent to the Carlitz identity, $F_{n+1}L_{n+2} - F_{n+2}L_n = F_{2n+1}$, shown in [3].

Example 2.2 Let us consider A_2, the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_2 = \{p_1 = p, p_2 = 1\}$. Then (8) tell us that $\{a_n\} \in A_2$ satisfies

$$a_n = \frac{2a_1 - (p - \sqrt{4+p^2})a_0}{2\sqrt{4+p^2}}\alpha^n - \frac{2a_1 - (p + \sqrt{4+p^2})a_0}{2\sqrt{4+p^2}}\left(-\frac{1}{\alpha}\right)^n, \tag{11}$$

where α is defined by

$$\alpha = \frac{p+\sqrt{4+p^2}}{2} \text{ and } \beta = -\frac{1}{\alpha} = \frac{p-\sqrt{4+p^2}}{2}. \tag{12}$$

Similarly, let $E_2 = \{1, q\}$. Then

$$a_n = \begin{cases} \frac{2a_1-(1-\sqrt{1+4q})a_0}{2\sqrt{1+4q}}\alpha_1^n - \frac{2a_1-(1+\sqrt{1+4q})a_0}{2\sqrt{1+4q}}\alpha_2^n, & if\ q \neq -\frac{1}{4}; \\ \frac{1}{2^n}(2na_1 - (n-1)a_0), & if\ q = -\frac{1}{4}, \end{cases}$$

where $\alpha = \frac{1}{2}(1+\sqrt{1+4q})$ and $\beta = \frac{1}{2}(1-\sqrt{1+4q})$ are solutions of equation $x^2 - x - q = 0$. The first special case (11) was studied by Falbo in [5]. If $p = 1$, the sequence is clearly the Fibonacci sequence. If $p = 2$ ($q = 1$), the corresponding sequence is the sequence of numerators (when two initial conditions are 1 and 3) or denominators (when two initial conditions are 1 and 2) of the convergent of a continued fraction to $\sqrt{2}$: $\{\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29} \ldots\}$, called the closest rational approximation sequence to $\sqrt{2}$. The second special case is for the case of $q = 2$ ($p = 1$), the resulting $\{a_n\}$ is the Jacobsthal type sequences (See Bergum, Bennett, Horadam, and Moore [2]).

From Proposition 2.2, for $E_2 = \{p, 1\}$, the Lucas sequence of A_2 with respect to E_2 is

$$\tilde{F}_n = \frac{1}{\sqrt{4+p^4}} \left\{ \left(\frac{p + \sqrt{4+p^2}}{2} \right)^n - \left(\frac{p - \sqrt{4+p^2}}{2} \right)^n \right\}.$$

In particular, the Lucas sequence for $E_2 = \{2, 1\}$ is the well-known Pell number sequence $\{P_n\} = \{0, 1, 2, 5, 12, 29, \ldots\}$ with the expression

$$P_n = \frac{1}{2\sqrt{2}} \left\{ (1+\sqrt{2})^n - (1-\sqrt{2})^n \right\}.$$

The Pell-Lucas number sequence, denoted by $\{q_n\}_{n\geq 0}$, is the sequence in A_2 with respect to $E_2 = \{2, 1\}$ and initial vector $(q_0, q_1) = (2, 1)$, which has the first few elements as $\{2, 1, 4, 9, 22, \ldots\}$. From (6) and (7), we obtain

$$P_n = \frac{3}{14} q_{n+1} + \frac{1}{14} q_{n-1}, \quad n \geq 1. \tag{13}$$

Similarly, for $E_2 = \{1, q\}$, the Lucas sequence of A_2 with respect to E_2 is

$$\tilde{F}_n = \frac{1}{\sqrt{1+4q}} \left\{ \left(\frac{1 + \sqrt{1+4q}}{2} \right)^n - \left(\frac{1 - \sqrt{1+4q}}{2} \right)^n \right\}.$$

In particular, the ISR for $E_2 = \{1, 2\}$ is the well-known Jacobsthal number sequence $\{J_n\} = \{0, 1, 1, 3, 5, 11, 21, \ldots\}$ with the expression

$$J_n = \frac{1}{3} \left(2^n - (-1)^n \right).$$

The Jacobsthal-Lucas number $\{j_n\}$ in A_2 with respect to $E_2 = \{1, 2\}$ satisfying $j_0 = 2$ and $j_1 = 1$ has the first few elements as $\{2, 1, 5, 7, 17, 31, \ldots\}$. From (5), one may have

$$j_n = J_n + 4J_{n-1} = 2^n + (-1)^n.$$

In addition, the above formula can transform all identities of Jacobsthal-Lucas number sequence to those of Jacobsthal number sequence. For example, we have

$$J_n^2 + 4J_{n-1}J_n = J_{2n},$$
$$J_m J_{n-1} - J_n J_{m-1} = (-1)^n 2^{n-1} J_{m-n},$$
$$J_m J_n + 2J_m J_{n-1} + 2J_n J_{m-1} = J_{m+n}$$

from

$$j_n J_n = J_{2n},$$
$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n},$$
$$J_m j_n - J_n j_m = 2J_{m+n},$$

respectively. Similarly, we can show that the following two identities are equivalent:

$$j_n = J_{n+1} + 2J_{n-1}, \quad J_{n+1} = J_n + 2J_{n-1}.$$

Furthermore, using (6) and (7), one may has

$$J_n = \frac{1}{9} j_{n+1} + \frac{2}{9} j_{n-1}, \quad n \geq 1, \tag{14}$$

which can be used to transform all identities of Jacobsthal number sequence to those of Jacobsthal-Lucas number sequence.

Remark 2.1 Proposition 2.2 can be extended to the linear nonhomogeneous recurrence relations of order 2 with the form: $a_n = pa_{n-1} + qa_{n-2} + \ell$ for $p + q \neq 1$. It can be seen that the above recurrence relation is equivalent to the homogeneous form (1) $b_n = pb_{n-1} + qb_{n-2}$, where $b_n = a_n - k$ and $k = \frac{\ell}{1-p-q}$. More details can be found in [6].

Example 2.3 An obvious example of Remark 2.1 is the Mersenne number $M_n = 2^n - 1$ $(n \geq 0)$, which satisfies the linear recurrence relation of order 2: $M_n = 3M_{n-1} - 2M_{n-2}$ (with $M_0 = 0$ and $M_1 = 1$) and the non-homogeneous recurrence relation of order 1: $M_n = 2M_{n-1} + 1$ (with $M_0 = 0$). It is easy to check that sequence $M_n = (k^n - 1)/(k - 1)$ satisfies both the homogeneous recurrence relation of order 2, $M_n = (k + 1)M_{n-1} - kM_{n-2}$, and the non-homogeneous recurrence relation of order 1, $M_n = kM_{n-1} + 1$, where $M_0 = 0$ and $M_1 = 1$. Here, M_n is the Lucas sequence with respect to $E_2 = \{3, -2\}$. Another example is Pell number sequence that satisfies both homogeneous recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ and the non-homogeneous relation $\bar{P}_n = 2\bar{P}_{n-1} + \bar{P}_{n-2} + 1$, where $P_n = \bar{P}_n + 1/2$.

Remark 2.2 In [11], Niven, Zuckerman, and Montgomery studied some properties of $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ defined respectively by the linear recurrence relations of order 2:

$$G_n = pG_{n-1} + qG_{n-2} \quad and \quad H_n = pH_{n-1} + qH_{n-2}$$

with initial conditions $G_0 = 0$ and $G_1 = 1$ and $H_0 = 2$ and $H_1 = p$, respectively. Clearly, $G_n = \tilde{F}_n$, the Lucas sequence of A_2 with respect to $E_2 = \{p_1 = p, p_2 = q\}$. Using Proposition 2.2, we may rebuild the relationship between the sequences $\{G_n\}$ and $\{H_n\}$:

$$\begin{aligned} H_n &= pG_n + 2qG_{n-1}, \\ G_n &= \frac{q}{p^2+4q} H_{n-1} + \frac{1}{p^2+4q} H_{n+1}. \end{aligned}$$

3 A type of Identities of Lucas sequence in A_2

Let A_2 be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_2 = \{p_1 = p, p_2 = q\}$, and let $\tilde{F}$ be the Lucas sequence of A_2. Inspired by [9], we give a nonlinear combinatorial expression involving $\tilde{F}$ and a numerous identities based on the expression. Using the interrelationship between the Lucas sequence and a linear recurring sequence in A_2, one may obtain many identities involving sequences in A_2. More precisely, let us consider the following extension of the results in [9] for the Fibonacci numbers to the general number sequences in A_2. Suppose $\{a_n\}_{n\in\mathbb{N}}$ be a nonzero sequence defined by the recurrence relation

$$a_n = p_1 a_{n-1} + p_2 a_{n-2}, \quad n \geq 2, p_1, p_2 \neq 0, \tag{15}$$

with the initial conditions $a_0 = 0$ and any nonzero a_1. Here, a_1 must be nonzero, otherwise $a_n \equiv 0$. Hence, we may normalize a_1 to be $a_1 = 1$ by define a new sequence $g_n = a_n/a_1$ satisfying the same recurrence relation (15). Thus, under the assumption, our sequence $\{a_n\}$ is the Lucas sequence $\{\tilde{F}_n\}$ of A_2 with respect to $E_2 = \{p_1, p_2\}$. We now give a nonlinear combinatorial expression involving $\tilde{F}_n$. Our result will extend to the case of $a_0 \neq 0$ and $a_1 = p_1 a_0$ later. In addition, sequence $\{\tilde{F}_n\}_{n\in\mathbb{N}}$ can be extended to the the case of $\{\tilde{F}_r\}_{r\in\mathbb{Z}}$ by using the same recurrence relation for $r \geq 1$ and $\tilde{F}_{r+1} = p_1\tilde{F}_r + p_2\tilde{F}_{r-1}$ while $r \leq -3$.

Lemma 3.1 *For any $m \in \mathbb{N}$ and $r \in \mathbb{Z}$ there holds*

$$\tilde{F}_{m+r} = \tilde{F}_m\tilde{F}_{r+1} + p_2\tilde{F}_{m-1}\tilde{F}_r. \tag{16}$$

Proof. For an arbitrarily $r \in \mathbb{Z}$, we have

$$\tilde{F}_{r+1} = \tilde{F}_1\tilde{F}_{r+1} + p_2\tilde{F}_0\tilde{F}_r$$

because $\tilde{F}_0 = 0$ and $\tilde{F}_1 = 1$. Assume (16) is true for $n \in \mathbb{N}$, $n \geq 1$, and an arbitrary $r \in \mathbb{Z}$, namely,

$$\tilde{F}_{r+n} = \tilde{F}_n\tilde{F}_{r+1} + p_2\tilde{F}_{n-1}\tilde{F}_r, \quad r \in \mathbb{Z}.$$

Then,

$$\tilde{F}_{r+n+1} = \tilde{F}_n\tilde{F}_{r+2} + p_2\tilde{F}_{n-1}\tilde{F}_{r+1}.$$

On the hand,

$$\begin{aligned}&\tilde{F}_{n+1}\tilde{F}_{r+1} + p_2\tilde{F}_n\tilde{F}_r = (p_1\tilde{F}_n + p_2\tilde{F}_{n-1})\tilde{F}_{r+1} + p_2\tilde{F}_n\tilde{F}_r\\ &= \tilde{F}_n\tilde{F}_{r+2} + p_2\tilde{F}_{n-1}\tilde{F}_{r+1},\end{aligned}$$

which implies

$$\tilde{F}_{r+n+1} = \tilde{F}_{n+1}\tilde{F}_{r+1} + p_2\tilde{F}_n\tilde{F}_r$$

and completes the proof with the mathematical indiction.

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A direct proof of (16) can also be given. Actually, every $\tilde{F}_m\tilde{F}_{r+1} + p_2\tilde{F}_{m-1}\tilde{F}_r$ can be reduced to $\tilde{F}_1\tilde{F}_{r+m} + p_2\tilde{F}_0\tilde{F}_r = \tilde{F}_{r+m}$ by using the recurrence relation (15).

Theorem 3.2 *For any given $m, n \in \mathbb{N}_0$ and $r \in \mathbb{Z}$ there holds*

$$\tilde{F}_{r+mn} = \sum_{j=0}^{n}\binom{n}{j}(\tilde{F}_m)^j(p_2\tilde{F}_{m-1})^{n-j}\tilde{F}_{r+j}. \tag{17}$$

Proof. Let $F(t) = \tilde{F}_{r+mt}$. Then from Lemma 3.1

$$\begin{aligned} & \Delta F(t) = F(t+1) - F(t) = \tilde{F}_{r+mt+m} - \tilde{F}_{r+mt} \\ = \ & \tilde{F}_m \tilde{F}_{r+mt+1} + (p_2 \tilde{F}_{m-1} - 1)\tilde{F}_{r+mt}. \end{aligned}$$

Thus, there holds symbolically

$$(\Delta - (p_2\tilde{F}_{m-1} - 1)I)\tilde{F}_{r+mt} = \tilde{F}_m \tilde{F}_{r+mt+1}.$$

Using the operator $\Delta - (p_2\tilde{F}_{m-1} - 1)I$ defined above j times, we find

$$(\Delta - (p_2\tilde{F}_{m-1} - 1)I)^j \tilde{F}_{r+mt} = (\tilde{F}_m)^j \tilde{F}_{r+mt+j}, \quad j \in \mathbb{N}.$$

Furthermore, noting the symbolic relation $E = I + \Delta$ and the last symbolical expression, one may find

$$\begin{aligned} & F(n) = \tilde{F}_{r+mn} = E^n \tilde{F}_{r+mt}\Big|_{t=0} = (I+\Delta)^n \tilde{F}_{r+mt}\Big|_{t=0} \\ = \ & (p_2\tilde{F}_{m-1} I + (\Delta - (p_2\tilde{F}_{m-1} - 1)I)^n \tilde{F}_{r+mt}\Big|_{t=0} \\ = \ & \sum_{j=0}^{n} \binom{n}{j} (p_2\tilde{F}_{m-1})^{n-j} (\Delta - (p_2\tilde{F}_{m-1} - 1)I)^j \tilde{F}_{r+mt}\Bigg|_{t=0} \\ = \ & \sum_{j=0}^{n} \binom{n}{j} (p_2\tilde{F}_{m-1})^{n-j} (\tilde{F}_m)^j \tilde{F}_{r+j} \end{aligned}$$

completing the proof of the theorem.

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Remark 3.1 The nonlinear expression for the case of $\{a_n\}$ with $a_0 = 0$ and $a_1 \neq 0$ can be specialized to the case $a_0 \neq 0$ and $a_1 = p_1 a_0$. We may normalize $a_0 = 1$ and define $a_{-1} = 0$ from the recurrence relation $a_1 = p_1 a_0 + p_2 a_{-1}$. Hence, the sequence $\{\hat{F}_n = a_{n-1}\}$ satisfies recurrence relation (15) for $n \geq 1$ with the initials $\hat{F}_0 = 0$ and $\hat{F}_1 = 1$. Hence, from (17) we have the nonlinear expression for $\hat{F}_n$ as

$$\hat{F}_{r+mn} = \sum_{j=0}^{n} \binom{n}{j} \hat{F}_{m-1}^k (q\hat{F}_{m-2})^{n-j} \hat{F}_{r+k}$$

for $m \geq 1$ and $r \geq 0$.

Similar to the last section and Remark 3.1, we may use the extension technique to define $\tilde{F}_n$ for negative integer index n. For example, substituting $n = 1$ into (15) yields $\tilde{F}_1 = p_1\tilde{F}_0 + p_2\tilde{F}_{-1}$, which defines $\tilde{F}_{-1} = 1/q$. With $r = -mn - 1$, $r = -mn$, and $r = -mn + 1$ in (17), a class of identities for $\tilde{F}_n$ with negative indices can be obtained as follows.

Corollary 3.3 *For $m \geq 1$ and $n \geq 0$ there hold the identities*

$$\begin{aligned}
&\sum_{j=0}^{n} p_2^{n-j+1}\binom{n}{j}(\tilde{F}_m)^j(\tilde{F}_{m-1})^{n-j}\tilde{F}_{j-mn-1} = 1,\\
&\sum_{j=0}^{n} p_2^{n-j}\binom{n}{j}(\tilde{F}_m)^j(\tilde{F}_{m-1})^{n-j}\tilde{F}_{j-mn} = 0,\\
&\sum_{j=0}^{n} p_2^{n-j}\binom{n}{j}(\tilde{F}_m)^j(\tilde{F}_{m-1})^{n-j}\tilde{F}_{j-mn+1} = 1.
\end{aligned} \tag{18}$$

Similarly, substituting $m = 2, 3$, and 4 into (17) and noting $\tilde{F}_2 = p$, $\tilde{F}_3 = p^2 + q$, and $\tilde{F}_4 = p(p^2 + 2q)$, we have

Corollary 3.4 *For $n \geq 0$, there hold identities*

$$\begin{aligned}
&\sum_{j=0}^{n} p_1^j p_2^{n-j}\binom{n}{j}\tilde{F}_{r+j} = \tilde{F}_{r+2n},\\
&\sum_{j=0}^{n} (p_1^2 + p_2)^j (p_1p_2)^{n-j}\binom{n}{j}\tilde{F}_{r+j} = \tilde{F}_{r+3n},\\
&\sum_{j=0}^{n} p_1^j p_2^{n-j}(p_1^2 + 2p_2)^j(p_1^2 + p_2)^{n-j}\binom{n}{j}\tilde{F}_{r+j} = \tilde{F}_{r+4n}.
\end{aligned} \tag{19}$$

With an application of Proposition 2.2, one may transfer the nonlinear expression (17) and its consequent identities shown in corollaries 3.3 and 3.4 to any linear recurring sequence defined by (1). For instance, from Corollary 3.4, we immediately have

Corollary 3.5 *Let us consider A_2, the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_2=\{p,q\}$. Then, for any $\{a_n\}\in A_2$, there hold*

$$\sum_{j=0}^{n} p_1^j p_2^{n-j}\binom{n}{j}(ca_{r+j-1}+da_{r+j-2}) = ca_{r+2n-1}+da_{r+2n-2},$$
$$\sum_{j=0}^{n}(p_1^2+p_2)^j(p_1p_2)^{n-j}\binom{n}{j}(ca_{r+j-1}+da_{r+j-2}) = ca_{r+3n-1}+da_{r+3n-2},$$
$$\sum_{j=0}^{n} p_1^j p_2^{n-j}(p_1^2+2p_2)^j(p_1^2+p_2)^{n-j}\binom{n}{j}$$
$$\times(ca_{r+j-1}+da_{r+j-2}) = ca_{r+4n-1}+da_{r+4n-2},$$

for $n\geq 0$, where c and d are given by

$$c=\frac{a_1-a_0p_1}{p_1(\tilde{F}_1-a_0a_1p_1-\tilde{F}_0p_2)},\quad d=-\frac{a_1p_2}{p_1(\tilde{F}_1-a_0a_1p_1-\tilde{F}_0p_2)},$$

provided that $p_1\neq 0$, and $\tilde{F}_1-a_0a_1p_1-\tilde{F}_0p_2\neq 0$.

The nonlinear expression (17) can be used to obtain a congruence relations involving products of the Lucas sequences as modules.

Corollary 3.6 *For $r\in\mathbb{Z}$, $m\geq 1$, and $n\geq 0$, there holds a congruence relation of the form*

$$\tilde{F}_{mn+r}\equiv (p_2\tilde{F}_{m-1})^n\tilde{F}_r+(\tilde{F}_m)^n\tilde{F}_{n+r}\ (mod\ \tilde{F}_{m-1}\tilde{F}_m). \tag{20}$$

In particular, for $r=0$ and $gcd\,(\tilde{F}_m,\tilde{F}_n)=1$,

$$\tilde{F}_{mn}\equiv 0\ (mod\ \tilde{F}_m\tilde{F}_n). \tag{21}$$

In general, if $\tilde{F}_{m_1}$, $\tilde{F}_{m_2},\ldots,$ $\tilde{F}_{m_s}$ be relatively prime to each other with each $m_k\geq 1$ $(k=1,2,\ldots,s)$, then there holds

$$\tilde{F}_{m_1m_2\cdots m_s}\equiv 0\ (mod\ \tilde{F}_{m_1}\tilde{F}_{m_2}\cdots\tilde{F}_{m_s}). \tag{22}$$

Proof. (20) comes from (17) straightforward. By setting $r = 0$, we have

$$\tilde{F}_{mn} \equiv (\tilde{F}_m)^n \tilde{F}_n \ (mod\ \tilde{F}_{m-1}\tilde{F}_m) \equiv 0 \ (mod\ \tilde{F}_m).$$

Similarly,

$$\tilde{F}_{mn} \equiv 0 \ (mod\ \tilde{F}_n).$$

Thus, if $gcd(\tilde{F}_m, \tilde{F}_n) = 1$, i.e., $\tilde{F}_m$ and $\tilde{F}_n$ are relatively prime, then we obtain (21), which implies (22).

■

Example 3.1 For $E_2 = \{1,1\}, \{1,2\}, and \{2,1\}$, formula (17) in Theorem 3.2 leads the following three non-linear identities for Fibonacci, Pell, and Jacobsthal number sequences, respectively:

$$F_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} F_m^j F_{m-1}^{n-j} F_{r+j},$$
$$P_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} P_m^j P_{m-1}^{n-j} P_{r+j},$$
$$J_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} J_m^j (2J_{m-1})^{n-j} J_{r+j},$$

where the first one is given in the main theorem of [9].

Example 3.2 As what we have presented, one may extend Fibonacci, Pell, and Jacobsthal numbers to negative indices as $\{F_n\}_{n\in\mathbb{Z}} = \{\ldots, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots\}$, $\{P_n\}_{n\in\mathbb{Z}} = \{\ldots, 5, -2, 1, 0, 1, 2, 5, 12, 29, \ldots\}$, and $\{J_n\}_{n\in\mathbb{Z}} = \{\ldots, 3/8, -1/4.1/2, 0, 1, 1, 3, 5, 11, \ldots\}$ by using the corresponding linear recurrence relation with respect to $E_2 = \{1, 1, \}, \{2, 1\}$, and $\{1, 2\}$, respectively. Thus, from the first formula of (18) in Corollary 3.3, there hold

$$\sum_{j=0}^{n}\binom{n}{j}F_m^jF_{m-1}^{n-j}F_{j-mn-1}=1,$$
$$\sum_{j=0}^{n}\binom{n}{j}P_m^jP_{m-1}^{n-j}P_{j-mn-1}=1,$$
$$\sum_{j=0}^{n}2^{n-j+1}\binom{n}{j}J_m^jJ_{m-1}^{n-j}J_{j-mn-1}=1.$$

The identities generated by using the other formulas in (18) and the formulas in Corollaries 3.4-3.6 can be written similarly, which are omitted here.

Example 3.3 By using the transformation formulas (10), (13), and (14), we may transform the nonlinear expressions shown in Examples 3.1 and 3.2 to those of the sequences in their sets with the same E_2 and initial vector $(a_0, a_1) = (2, 1)$, respectively. For instance, from Example 3.1, there hold

$$\begin{aligned}
&L_{mn+r+1}+L_{mn+r-1}\\
=\;&\frac{1}{5^n}\sum_{j=0}^{n}\binom{n}{j}(L_{m+1}+L_{m-1})^j(L_m+L_{m-2})^{n-j}(L_{r+j+1}+L_{r+j-1}),\\
&q_{mn+r+1}+q_{mn+r-1}\\
=\;&\frac{1}{14^n}\sum_{j=0}^{n}\binom{n}{j}(3q_{m+1}+q_{m-1})^j(3q_m+q_{m-2})^{n-j}(3q_{r+j+1}+q_{r+j-1}),\\
&j_{mn+r+1}+j_{mn+r-1}\\
=\;&\left(\frac{2}{9}\right)^n\sum_{j=0}^{n}\binom{n}{j}\frac{1}{2^j}(j_{m+1}+2j_{m-1})^j(j_m+2j_{m-2})^{n-j}(j_{r+j+1}+2j_{r+j-1}).
\end{aligned}$$

Similarly, from Example 3.2, we have

$$\frac{1}{5^{n+1}}\sum_{j=0}^{n}\binom{n}{j}(L_{m+1}+L_{m-1})^j(L_m+L_{m-2})^{n-j}$$
$$\times(L_{j-mn}+L_{j-mn-2})=1$$
$$\frac{1}{14^{n+1}}\sum_{j=0}^{n}\binom{n}{j}(3q_{m+1}+q_{m-1})^j(3q_m+q_{m-2})^{n-j}$$
$$\times(3q_{j-mn}+q_{j-mn-2})=1$$
$$\frac{1}{9^{n+1}}\sum_{j=0}^{n}2^{n-j+1}\binom{n}{j}(j_{m+1}+2j_{m-1})^j(j_m+2j_{m-2})^{n-j}$$
$$\times(j_{j-mn}+2j_{j-mn-2})=1$$

At the end of this section, we will mentioned a relationship, established in [7] (inspired by Aharonov, Beardon, and Driver[1]) by Shiue, Weng and the author, between the recurring numbers defined by (1) and the values of the Gegenbauer-Humbert polynomials including the Chebyshev polynomials of the second kind, $U_n(x)$, and the Fibonacci polynomials, $\bar{F}_n(x)$. From Corollary 2.2 of [7], we have the relationships

$$\tilde{F}_n = \left(\sqrt{-p_2}\right)^{n-1} U_{n-1}\left(\frac{p_1}{2\sqrt{-p_2}}\right),$$
$$\tilde{F}_n = \left(\sqrt{p_2}\right)^{n-1} \bar{F}_n\left(\frac{p_1}{\sqrt{p_2}}\right),$$
$$\tilde{F}_n = \left(-\sqrt{-p_2}\right)^{n-1} U_{n-1}\left(\frac{-p_1}{2\sqrt{-p_2}}\right),$$
$$\tilde{F}_n = \left(-\sqrt{p_2}\right)^{n-1} \bar{F}_n\left(\frac{-p_1}{\sqrt{p_2}}\right).$$

In particular, for $E_2(1,1)$, the above relationships present the expressions of Fibonacci numbers in term of the values of the Chebyshev polynomials of the second kind and the Fibonacci polynomials as follows:

$$\begin{aligned} F_n &= i^{n-1} U_{n-1}\left(-\frac{i}{2}\right), \\ F_n &= \bar{F}_n(1), \\ F_n &= (-i)^{n-1} U_{n-1}\left(\frac{i}{2}\right), \\ F_n &= (-1)^{n-1} \bar{F}_n(-1), \end{aligned}$$

where the first formula can be seen in [1]. Similarly, for $E_2 = (2,1)$, we have

$$\begin{aligned} P_n &= i^{n-1} U_{n-1}(-i), \\ P_n &= \bar{F}_n(2), \\ P_n &= (-i)^{n-1} U_{n-1}(i), \\ P_n &= (-1)^{n-1} \bar{F}_n(-2). \end{aligned}$$

If $E_2 = (1,2)$, then the relationships between the Jacobsthal numbers and the values of the Chebyshev polynomials of the second kind and the Fibonacci polynomials are

$$\begin{aligned} J_n &= \left(\sqrt{2} i\right)^{n-1} U_{n-1}\left(-\frac{i}{2\sqrt{2}}\right), \\ J_n &= 2^{(n-1)/2} \bar{F}_n\left(\frac{1}{\sqrt{2}}\right), \\ J_n &= \left(-\sqrt{2} i\right)^{n-1} U_{n-1}\left(\frac{i}{2\sqrt{2}}\right), \\ J_n &= \left(-\sqrt{2}\right)^{n-1} \bar{F}_n\left(\frac{-1}{\sqrt{2}}\right). \end{aligned}$$

Example 3.4 Using the above relationships, we may change the non-linear expressions of $\tilde{F}_n$ to the non-linear expressions for the values of the Chebyshev polynomials of the second kind and the Fibonacci polynomials, respectively. For instance, from Example 3.1, there hold

$$U_{mn+r-1}\left(-\frac{i}{2}\right)=$$
$$\sum_{j=0}^{n}\binom{n}{j}i^{-2(n-j)}U_{m-1}\left(-\frac{i}{2}\right)^{j}U_{m-2}\left(-\frac{i}{2}\right)^{n-j}U_{r+j-1}\left(-\frac{i}{2}\right)$$
$$U_{mn+r-1}(-i)=\sum_{j=0}^{n}\binom{n}{j}i^{-2(n-j)}U_{m-1}\left(-i\right)^{j}U_{m-2}\left(-i\right)^{n-j}U_{r+j-1}\left(-i\right)$$
$$U_{mn+r-1}\left(-\frac{i}{2\sqrt{2}}\right)=$$
$$\sum_{j=0}^{n}\binom{n}{j}(i)^{-2(n-j)}U_{m-1}\left(-\frac{i}{2\sqrt{2}}\right)^{j}U_{m-2}\left(-\frac{i}{2\sqrt{2}}\right)^{n-j}U_{r+j-1}\left(-\frac{i}{2\sqrt{2}}\right).$$

Other nonlinear expressions of the values of the Chebyshev polynomials of the second kind and the Fibonacci polynomials can be constructed similarly from Examples 3.1 and 3.2, which we omitted here.

References

[1] D. Aharonov, A. Beardon, and K. Driver, Fibonacci, Chebyshev, and orthogonal polynomials, Amer. Math. Monthly. 122 (2005), 612–630.

[2] G. E. Bergum, L. Bennett, A. F. Horadam, and S. D. Moore, Jacobsthal Polynomials and a Conjecture Concerning Fibonacci-Like Matrices, Fibonacci Quart. 23 (1985), 240-248.

[3] L. Carlitz, Problem B-110, Fibonacci Quart., 5 (1967), No. 5, 469-470.

[4] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.

[5] C. Falbo, The golden ratio - a contrary viewpoint, The College Math J. 36 (2005), No. 2, 123–134.

[6] T. X. He and P. J.-S. Shiue, On sequences of numbers and polynomials defined by second order recurrence relations, Internat. J. Math. and Math. Sci., Volume 2009 (2009), Article ID 709386, 1-21.

[7] T. X. He, P. J.-S. Shiue, and T. W. Weng, Hyperbolic expressions of polynomial sequences defined by linear recurrence relations of order 2, ISRN Discrete Math., Volume 2011, Article ID 674167, 16 pages, doi:10.5402/2011/ 674167.

[8] L. C. Hsu, Computational Combinatorics (Chinese), First edition, Shanghai Scientific & Techincal Publishers, Shanghai, 1983.

[9] L. C. Hsu, A nonlinear expression for Fibonacci numbers and its consequences, Journal of Mathematical Research and Applications, 32 (2012), 654-658.

[10] T. Koshy, Fibonacci and Lucas numbers with applications, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001.

[11] I. Niven, H. S. Zuckerman, and H. L. Montgomery, An introduction to the theory of numbers, Fifth edition, John Wiley & Sons, Inc., New York, 1991.

[12] G. Strang, Linear algebra and its applications. Second edition. Academic Press (Harcourt Brace Jovanovich, Publishers), New York-London, 1980.

[13] H. S. Wilf, Generatingfunctionology, Academic Press, New York, 1990.