Composite Dilation Wavelets with High Degrees

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Construction of multivariate wavelets on arbitrary triangulation

- Haar-type non-separable constant wavelets: "twin dragon," Belogay and Wang [99], Flaherty and Wang [99], and Gröchenig and Madych [92]; wavelet with composite dilations, Guo, Kutyniok, and Labate [04], Krishtal, Robinson, Weiss, and Wilson [08], Krommweh [09], Krommwek and Plonka [09], Blanchard [09], MacArthur and Taylor [11], Blanchard and Krishtal [12], Grohs [13].
- Continuous piecewise linear wavelet: Yserenntant [86], Vassilevski and Wang [97], Stevenson [97, 98], Liu [06], Floater and Quak [99, 00], Hardin and Hong [03].
- C^1 quadratic splines and spline wavelets: Powell [73], Powell-Sabin [77], Chui and He[90], Chui, Chui, and He[93], Chui and Jiang [04]; Oswald [92], Davydov and Petrushev [03,05], Maes and Bultheel [07, 08, 09, 10], Windmolders, Vanraes, Dierckx, and Bultheel [03], Maes, Vanraes, Dierckx, and Bultheel [04], Speleers, Dierckx, and Vandewalle [06, 07, 08, 09]

Splines and elements, spline wavelets, wavelets with composite dilations

- Splines and their BB-expressions: Farin [88,90,93], Chui [87], etc.
- Characterization of compactly supported refinable splines: Lawton, Lee, and Shen [95], Sun [96], Goodman [98], Guan and He [09], etc.
- Spline wavelets: Chui and Wang [92,93], Chui, Stökler, and Ward [92], Jia and Micchelli [91], Riemenschneider and Shen [92], Lorentz and Oswald [00, Sobolev spaces], Jia, Wang, and Zhou [03], Jia and Liu [08], etc.
- Wavelets with composite dilations: Guo, Labate, Lim, Weiss, and Wilson [04, 06, 06], Guo and Labate [07, 08,10, 11,12,13], Guo and Labate and Lim [09], etc.

A function f supported in $[0,1]^2$ wish a discontinuity across a nice curve Γ and otherwise smoothness.

$\tilde{f}_m^F$ standard Fourier approximation built from the best m nonzero Fourier terms.

$$\|f-\tilde{f}_m^F\|_2^2 \leq cm^{-1/2}, \quad m\to\infty.$$

$\tilde{f}_m^W$ wavelet non-adaptive approximation built from the best m nonzero wavelet terms.

$$\|f-\tilde{f}_m^W\|_2^2 \leq cm^{-1}, \quad m\to\infty.$$

$\tilde{f}_m^A$ wavelet adaptive approximation built from the best m nonzero wavelet terms.

$$\|f-\tilde{f}_m^A\|_2^2 \leq cm^{-2}, \quad m\to\infty.$$

$\tilde{f}_m^C$ (non-adaptive) curvelet m-term approximation-summing the m biggest terms in the curvelet frame expansion.

$$\|f-\tilde{f}_m^C\|_2^2 \leq Cm^{-2}(\log m)^3, \quad m\to\infty.$$

Composite dilations: Curvelets (Candés and Donoho [03])

$$\psi_{a,b,\theta}(x) = a^{-\frac{3}{4}}\psi(D_a R_\theta(x-b)),$$

$$D_a = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}, R_\theta = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}, \quad R_\theta^{-1} = R_\theta^T = R_{-\theta}$$

$$c(a,b,\theta) = \langle f, \psi_{a,b,\theta}\rangle = \int_{R^2} f(x)\overline{\psi_{a,b,\theta}(x)}dx$$

Define ψ_j by $\hat{\psi}_j(\xi) = \cup_j(\xi) := 2^{-\frac{3j}{4}} W(2^{-j}\gamma)V(\frac{2^{\lfloor j/2 \rfloor}\theta}{2\pi})$ with $\sum_{j=\infty}^{\infty} W^2(2^j\gamma) = 1$ and $\sum_{l=-\infty}^{\infty} V^2(t-l) = 1$. When $r \in (\frac{3}{4}, \frac{3}{2})$ and $t \in (-\frac{1}{2}, \frac{1}{2})$

$$\psi_{j,l,k} = \psi_j(R_{\theta_l}(x - x_k^{j,l}) \quad f = \sum_{j,l,k}\langle f, \psi_{j,l,k}\rangle\psi_{j,l,k}$$

$$\|f\|^2 = \sum_{j,l,k} |\langle f, \psi_{j,l,k}\rangle|^2$$

Composite dilations: Shearlets (Krishtal, Robinson, Weiss, and Wilson [08])

Let $A_{ast}(\psi) := \{\psi_{ast}(x) = a^{-\frac{3}{4}}\psi(M_{as}^{-1}(x-t)) : a \in R^{+}, s \in R,$ $t \in R^2\}$, where $M_{as} := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} = \begin{pmatrix} a & \sqrt{a}s \\ 0 & \sqrt{a} \end{pmatrix}$ We have

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_2}{\xi_1}),$$

where $\hat{\psi}_1 \in C^{\infty}(\mathbb{R})$ with *supp* $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and $\hat{\psi}_2 \in C^{\infty}(\mathbb{R})$ with *supp* $\hat{\psi}_2 \subset [-1, 1], \hat{\psi}_2 > 0$ on $(-1, 1)$, and $\|\psi_2\| = 1$. The family $\{\psi_{ast}(x)\}$ is a reproducing system for $L^2(\mathbb{R}^2)$, i.e., it satisfies the Calderson's formula

$$\|f\|^2 = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^{\infty} |\langle f, \psi_{ast}\rangle|^2 \frac{da}{a^3} dsdt$$

for all $f \in L^2(\mathbb{R}^2)$.

Wavelets with composite dilations-Shearlets-2

Continuous Shearlet transform

$$S_f(a,s,t) = \langle f, \psi_{ast} \rangle, a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2$$

The continuos Shearlet are not only able to locate a discontinuity curve, but also to identify its orientation. That is, for $a \to 0$, the Shearlet transforms $S_f(a,s,t)$ tends to 0 rapidly unless t is at the singularity and s describe the direction that is perpendicular to the discontinuity curve. Thus

$$\hat{\psi_{ast}} = a^{-\frac{3}{4}} e^{-2\pi i \xi t} \hat{\psi_1}(a\xi_1) \hat{\psi_2}(a^{-\frac{1}{2}}(s + \frac{\xi_2}{\xi_1}))$$

where
$supp\ \hat{\psi_{ast}} \subset \{(\xi_1, \xi_2) : \xi_1 \in [-\frac{2}{a}, -\frac{1}{2a}] \cup [\frac{1}{2a}, \frac{2}{a}], |s + \frac{\xi_2}{\xi_1}| \leq \sqrt{a}\}$.

Wavelets with composite dilations-Shearlets-3

Ex. Let $f = X_D$, D is the unit disc in $\mathbb{R}^2$ for $a \to 0$ if $t \in \partial D$ and s describes the direction normal to ∂D, then $|S_f(a,s,t)| \le ca^{\frac{3}{4}}$. Otherwise, for each $N = 1, 2, ..$ we have $|S_f(a,s,t)| \le ca^N$. Let $a = 2$, denote M_{2s} by $M_{ij} = \begin{pmatrix} 2^i & 0 \\ 0 & 2^{\frac{i}{2}} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$. Then $A_{ijt}(\psi)$ gives a discrete system. Define $\psi \in L^2(R^2)$ by $\hat{\psi} = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_1}{\xi_2})$ where $\hat{\psi}_1 \in L^2(\mathbb{R})$ with *supp* $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and $\hat{\psi}_2 \in L^2(\mathbb{R})$ with *supp* $\hat{\psi}_2 \subset [-1, 1]$ satisfying

$$\sum_{j \in Z} |\hat{\psi}_2(\xi + j)|^2 = 1 \text{ and } \sum_{j \in Z} |\hat{\psi}_1(2^j \xi)|^2 = 1 \quad a.e \xi \in \mathbb{R}.$$

Thus $\hat{\psi} \in C^\infty(\mathbb{R}^2)$, $|\psi(x)| \le k_N(1 + |x|)^{-N}$ where $k_N > 0$ for any $N \in \mathbb{N}$. In addition, $\sum_{i,j,k} |\langle f, \psi_{i,j,k}\rangle|^2 = \|f\|^2$ for all $f \in L^2(R^2)$.

Wavelets with composite dilations-AB-MRA-1

Let $A \in GL_2(\mathbb{R})$ and
$\{B_j : j \in \mathbb{Z}\} \subset \tilde{S}L_2(\mathbb{Z}) = \{b \in GL_n(\mathbb{R}) : |det\ b| = 1\}$.

(i) $D_{B_j} T_k V_0 = V_0$ for any $j \in \mathbb{Z}, k \in \mathbb{Z}^2$.

(ii) For each $i \in \mathbb{Z}$, $V_i \subset V_{i+1}$, where $V_i = D_{A^{-i}} V_0$.

(iii) $\cap_i V_i = \{0\}$ and $\overline{\cup_i V_i} = L^2(\mathbb{R}^2)$.

(iv) $\exists\ \phi \in L^2(\mathbb{R}^2)$ such that $\Phi_B = \{D_{B_j} T_k \phi : j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$ is a tight frame (resp. ON basis) for V_0.

V_0: AB scaling space, ϕ: AB scaling function (resp. ON AB scaling function) for V_0. *AB*-MRA shearlet basis: $A_{AB}(\bar{\Psi}) = \{D_{A^i} D_{B_j} T_k \Psi : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}\}$, where $\bar{\psi} = (\psi^1, \psi^2, ..., \psi^L)$ and $\{B_j : j \in \mathbb{Z}, |detB_j| = 1\}$.

Wavelets with composite dilations-AB-MRA-2

Let $\bar{\psi} = (\psi^1, \psi^2, ..., \psi^L) \in L^2(\mathbb{R}^2)$ be such that $\{D_{B_j} T_k \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, \ldots, L\}$ is an ON basis (resp. tight frame) for W_0, the o.r. complement of V_0 in V_1. Then $\bar{\Psi}$ is an ON (resp. tight frame) AB-multiwavelet.

EX. Let $A := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, $B_j := B^j, j \in Z, B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$S_0 = \{\xi = (\xi_1, \xi_2) \in R^2 : |\xi_1| < \frac{1}{4}\}$ is the vertical strip of width $\frac{1}{2}$ bounded by the lines $\xi_1 = \pm\frac{1}{4}$.

$$S_i = A^i S_0, i \in Z, \{\xi = (\xi_1, \xi_2) : |\xi_1| < 2^{2i-2}\}$$

$$(B^T)^j \xi = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ j\xi_1 + \xi_2 \end{pmatrix} \Rightarrow (B^T)^j S_0 \subseteq S_0, j \in Z$$

We have $(i) S_i \subseteq S_{i+1}, (ii) \cup_{i \in Z} S_i = R^2, (iii) \cap_{i \in Z} S_i = \{\xi = (\xi_1, \xi_2) \in R^2, \xi_1 = 0\}$.

Wavelets with composite dilations-AB-MRA-3

Define $L^2(S) = \{f \in L^2(R^2) : supp\ \hat{f} \subseteq S\}$.

(i) $D^j_{B^T} T_k L^2(S_0) = L^2(S_0)$, for any $j \in Z, k \in Z^2$.

(ii) $L^2(S_i) \subseteq L^2(S_{i+1})$

(iii) $\cap_{i \in Z} L^2(S_i) = 0$ and $\overline{\cup_{i \in Z} L^2(S_i)} = L^2(R^2)$.

Let $\hat{\phi} = X_U, U = U^+ \cup U^-, U^+$ is the triangle with vertices $(0,0), (\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4})$ and $U^- = -U^+$. Thus $S_0 = \cup_{i \in Z} (B^T)^j U$, where the union is disjoint. Hence, Φ_B is a tight frame for V_0. Therefore, $\{L^2(S_i) = V_i : i \in Z\}$ is a AB-MRA. AB-MRA shearlet: $R_0 := S_1 / S_0$. $W_0 = L^2(R_0)$ is the complement of V_0 in V_1. Set $I = I^+ \cup I^-$ in R_0, which is defined by I^+ is the trapezoid with vertices $(\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}), (1,0) \& (1,1) \& I^- = -I^+$. We have $\hat{\psi} = X_I . \{D_{B^j} T^k \psi : j \in Z, k \in Z^2\}$ is a tight frame for W_0 and ψ is a tight frame AB-wavelet.

Spline wavelets with composite dilations-1

Haar-type wavelets

$$D_c : (D_c f)(x) = |det\ c|^{-\frac{1}{2}} f(c^{-1}x)$$
$$\Gamma : \ a\ lattice\ in\ \mathbb{R}^d\ with\ \Gamma = M\mathbb{Z}^d.$$
$$T_\gamma : \gamma \in \Gamma \ \ (T_\gamma f)(x) = f(x-\gamma)$$
$$\Psi = \{\psi^1, \psi^2, ..., \psi^L\}, \psi^l \in L^2(\mathbb{R}^d), l = 1, ..., L$$

$\{D_c T_\gamma \psi^l : c \in C, \gamma \in \Gamma, l = 1, 2, ..., L\}$ is an orthonormal wavelet system or multiwavelet associated with dilation C and lattice Γ. Consider $C = AB, A, B \in GL_d(\mathbb{R}) : |det\ b| = 1$ for all $b \in B$ and $|det\ a| \leq 1$ for all $a \in A$. B is finite, $A(\Gamma) \subset \Gamma$ and $B(\Gamma) = \Gamma$.

Spline wavelets with composite dilations-2

MRA associated with a sequence of dilations $\{a^j\}_{j\in\mathbb{Z}} = A$, a is an expanding matrix, in an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(R^d)$ such that

(i) $V_j = D_{a^{-j}} V_0$

(ii) $\cap_{j\in Z} V_j = \{0\}$

(iii) $\overline{\cup_{j\in Z} V_j} = L^2(\mathbb{R}^d)$

(iv) $\exists \varphi \in V_0$ such that $\{T_\gamma \varphi\}, \gamma \in \Gamma$, is an orthonormal basis of V_0.

Since $B(\Gamma) = \Gamma$, the operators generated by the dilations $D_b, b \in B$, and the translations $T_\gamma, \gamma \in \Gamma$, form a group with operation $(c,\tau).(b,\gamma) = (cb, b^{-1}\tau + \gamma)$, i.e. $(D_c T_c)(D_b T_\gamma) f = D_{cb} T_{b^{-1}\tau+\gamma}$. $B\Gamma$ is a semi-direct product of $B\&\Gamma$. $B\Gamma$-invariant space $V \subset L^2(\mathbb{R}^d) : D_b T_\gamma f \in V$ for every $f \in V, b \in B$, and $\gamma \in \Gamma$. Thus, (iv)' $\varphi \in V_0$ such that $\{D_b T_\gamma \varphi : b \in B, \gamma \in \Gamma\}$ is an orthonormal basis for V_0. AB-MRA: (i)-(iii) and (iv)'

Spline wavelets with composite dilations-3

An example of compactly supported multiwavelet:
$\Psi = \{\psi^1, \psi^2, ...\psi^L\}$ such that the system
$\{D_{a^j} D_b T_\gamma \psi^l : j \in Z, b \in B, \gamma \in \Gamma, l = 1, .., L\}$ is an orthonormal basis for $L^2(R^d)$: Let $a = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} =: q$ be the expanding matrix, and let $B = \{b_i : i = 0, 1, ..., 7\}$ be the group of matrices

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} b_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} b_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b_i = -b_{i-4}, i = 4, 5, 6, 7$$

$R_i = b_i R_0$ for $i = 0, 1, .., 7.$
$\varphi := 2\sqrt{2} X_{R^0}$, $V_0 := \{D_b T_k \varphi : b \in B, k \in Z^2\}$ is an orthonormal basis for its closed linear span $V_j := D_{q^{-j}} V_0, j \in Z$.

Spline wavelets with composite dilations-4

Hence, V_0 is the subspace of $L^2(R^2)$ consisting of all square integrable functions that are constant on each Z^2-translate of the triangles $R_i, i = 0, 1, 2, .., 7$. V_1 consists of all functions in $L^2(R^2)$ that are constant on each $q^{-1}Z^2$-translate of the triangles $q^{-1}R_i, i = 0, 1, .., 7. V_0 \subset V_1$ and consequently, $V_j \subset V_{j+1}$ for all $j \in Z$. $V_j, j \in Z$ form an AB-MRA with φ as a scaling function. As we mentioned above, another point of view is to consider the column vector

$$\Phi = \begin{pmatrix} D_{b_0}\varphi \\ \vdots \\ D_{b_7}\varphi \end{pmatrix} = \begin{pmatrix} \varphi^0 \\ \vdots \\ \varphi^7 \end{pmatrix}$$

to be scaling function vector for this MRA.

$$R_0 = q^{-1}R_1 \cup \left[q^{-1}R_6 + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right] = q^{-1}R_1 \cup q^{-1} \left[R_6 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Spline wavelets with composite dilations-5

Hence, $X_{R_0}(x) = X_{q^{-1}R_1}(x) + X_{q^{-1}(R_6 + \begin{pmatrix}0\\1\end{pmatrix})}$ or equivalently

$\varphi^0(x) = \varphi^1(qx) + \varphi^6(qx - \begin{pmatrix}0\\1\end{pmatrix})$. Applying D_{b_i} to the above

$$\begin{cases}
\varphi^0(x) = \varphi^1(qx) + \varphi^6(qx - \begin{pmatrix}0\\1\end{pmatrix}), & \varphi^1(x) = \varphi^2(qx) + \varphi^5(qx - \begin{pmatrix}1\\0\end{pmatrix}) \\
\varphi^2(x) = \varphi^3(qx) + \varphi^0(qx + \begin{pmatrix}1\\0\end{pmatrix}), & \varphi^3(x) = \varphi^4(qx) + \varphi^7(qx + \begin{pmatrix}0\\1\end{pmatrix}) \\
\varphi^4(x) = \varphi^5(qx) + \varphi^2(qx + \begin{pmatrix}0\\1\end{pmatrix}), & \varphi^5(x) = \varphi^6(qx) + \varphi^1(qx + \begin{pmatrix}0\\1\end{pmatrix}) \\
\varphi^6(x) = \varphi^7(qx) + \varphi^4(qx - \begin{pmatrix}1\\0\end{pmatrix}), & \varphi^7(x) = \varphi^0(qx) + \varphi^3(qx - \begin{pmatrix}1\\0\end{pmatrix})
\end{cases}$$

Spline wavelets with composite dilations-6

$\psi^0(x) = \varphi^1(qx) - \varphi^6(qx - \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ is a Haar-type spline wavelet.

Moreover, the system $\{D_{q^i} D_b T_k \psi : j \in Z, b \in B, k \in Z^2\}$ is an orthonormal basis for $L^2(R^2)$. Since the matrices in B have the integer entries and $|det\ B| = 1, b \in B$,

$$\{D_{q^i} D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^2\}$$
$$= \{D_{q^i} T_k (D_b \psi) : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^2\},$$

where $\psi^i := D_{b_i} \psi, i = 0, 1, .., 7$. Thus, $(\psi^0, .., \psi^7)^T$ forms a Haar-type spline multiwavelet.

$$\hat{\Phi} := \begin{pmatrix} \hat{\varphi^0} \\ \vdots \\ \hat{\varphi^7} \end{pmatrix} \quad \hat{\Psi} := \begin{pmatrix} \hat{\psi^0} \\ \vdots \\ \hat{\psi^7} \end{pmatrix}$$

Spline wavelets with composite dilations-7

We have $\xi q = (\xi_1 + \xi_2, \xi_2 - \xi_1)$, and $\hat{\Phi}(\xi q) = M_0(\xi)\hat{\Phi}(\xi)$, where low-pass filter is

$$M_0(\xi)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & e(-\xi_2) & 0 \\ 0 & 0 & 1 & 0 & 0 & e(-\xi_2) & 0 & 0 \\ e(\xi_1) & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & e(\xi_1) \\ 0 & 0 & e(\xi_2) & 0 & 0 & 1 & 0 & 0 \\ 0 & e(\xi_2) & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e(-\xi_1) & 0 & 0 & 1 \\ 1 & 0 & 0 & e(-\xi_1) & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $e(\alpha) = e^{2\pi i \alpha}$ for $\alpha \in R$. The high-pass filter M_1 is $M_1(\xi) = M_0(\xi + \beta)$, $\beta = (\frac{1}{2}, \frac{1}{2})$ satisfying $M_0(\xi)M_0^*(\xi) + M_1(\xi)M_1^*(\xi) = I$, M^* is the conjugate transpose of M. Hence, $\hat{\psi}(\xi q) = M_1(\xi)\hat{\phi}(\xi)$.

BB-expressions of polynomials and splines-1

Let $x^0, \dots x^d \in \mathbb{R}^d$, $d \geq 1, x^i = (x_1^i, \dots, x_d^i)$ and consider the convex hull

$$T_d := \langle x^0, \dots, x^d \rangle = \{\sum_{i=0}^{d} \alpha_i x^i : \sum_{i=0}^{d} \alpha_i = 1, \alpha_i \geq 0\}.$$

This convex hull is called an *d-simplex* if its signed volume $Vol_d\langle x^0, \dots, x^d \rangle$ is nonzero. Suppose that $\langle x^0, \dots, x^d \rangle$ is an d-simplex. Then any $x \in \mathbb{R}^d$ can be identified by an $(d+1)$-tuple $\lambda = (\lambda_0, \dots, \lambda_d)$, the *barycentric coordinates* of x relative to the d-simplex $\langle x^0, \dots, x^d \rangle$, where

$$\lambda_i = \lambda_i(x) = \frac{Vol_d\langle x^0, \dots, x^{i-1}, x, x^{i+1}, \dots, x^d \rangle}{Vol_d\langle x^0, \dots, x^d \rangle}.$$

BB-expressions of polynomials and splines-2

Thus, each $\lambda_i = \lambda_i(x)$ is a linear polynomial in x with $\sum_{i=0}^d \lambda_i = 1$, and if $x \in \langle x^0, ..., x^d \rangle$, then $\lambda_i \geq 0$.
For any $b = (\beta_0, \ldots, \beta_d) \in \mathbb{Z}_+^{d+1}$, and $n \in \mathbb{Z}_+$, we will use the usual multivariate notation $\lambda^b = \lambda_0^{\beta_0} \cdots \lambda_d^{\beta_d}$, $b! = \beta_0! \cdots \beta_d!$, and $|b| = \beta_0 + \cdots + \beta_d$. Hence,

$$\phi_b^n(\lambda) := \frac{n!}{b!} \lambda^b \tag{1}$$

is a polynomial in $\pi_{|\beta|}^d$, the space of all polynomials in d variables of order $|\beta| + 1$, or degree at most $|\beta|$.

BB-expressions of polynomials and splines-3

With any set $\{a^n_\beta\} = \{a^n_\beta\}_{\beta\in\mathbb{Z}^{d+1}_+,|\beta|=n} \subset \mathbb{R}$ one may associate the polynomial

$$p_n(x) = B_n[\{a^n_\beta\};\lambda] = \sum_{|\beta|=n} a^n_\beta \phi^n_\beta(\lambda), \tag{2}$$

which is called a *Bernstein-Bézier polynomial (BB polynomial)* of total degree n relative to the d-simplex $\langle x^0,\ldots,x^d\rangle$,. In addition, $\{a^n_\beta : |\beta| = n\}$ shown as in (2) is called the set of *Bézier coefficients* of the polynomial p_n. The piecewise linear interpolant to the points $(\beta/n, a^n_\beta)$ is said to be the *Bézier net* or *control net*. and is displayed schematically in Figure 1 for the case of $n = 2$ and $d = 2$.

BB-expressions of polynomials and splines-4

Denote $D_y = \sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i}$, where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. For $y = x^i - x^j$, we denote

$$D_{ij} = D_y = D_{x^i - x^j}, \quad i \neq j.$$

By using the barycentric coordinates $\{\lambda_\ell\}_{\ell=0}^{d}$ of $x \in \mathbb{R}^d$ relative to an d-simplex $T_d = \langle x^0, ..., x^d \rangle$, we can write $x = \sum_{\ell=0}^{d} \lambda_\ell x^\ell$. If we define

$$E_i a_\alpha := a_{\alpha + \mathbf{e}^i}$$

and

$$\triangle_{ij} a_\alpha^n = E_i a_\alpha^n - E_j a_\alpha^n,$$

where $\mathbf{e}^i = (\delta_{ij})_{j=0}^{d}$ denotes the i^{th} coordinate vector in $\mathbb{R}^{d+1}$, then

$$D_{ij} p_n = n \sum_{|\alpha|=n-1} (E_i - E_j) a_\alpha^n \phi_\alpha^{n-1}(\lambda) = n \sum_{|\alpha|=n-1} \triangle_{ij} a_\alpha^n \phi_\alpha^{n-1}(\lambda). \tag{3}$$

Continuous Wavelets with Composite Dilation-1

Let B be the group of order 3 generated by the matrix $\rho = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$, which is the counter-clockwise rotation by $2\pi/3$. Consider the hexagon H centered at the origin consisting of the diamonds $R_i = (v_{i0}, v_{i1}, v_{i2}, v_{i3})$, $(i = 0, 1, 2)$, where v_{i0}, v_{i1}, v_{i2}, and v_{i3} are vertices of R_i, and R_0 has vertices $v_{00} = (0,0)$, $v_{01} = (\sqrt{3}/4, -3/4)$, $v_{02} = (\sqrt{3}/2, 0)$, $v_{03} = (\sqrt{3}/4, 3/4)$.

Continuous Wavelets with Composite Dilation-2

The elements of B map R_0 onto other diamonds $R_i = \rho^i R_0$ $(i = 1, 2)$. Let $C = \frac{1}{4}\begin{pmatrix} 0 & 3\sqrt{3} \\ 6 & 3 \end{pmatrix}$ and $\Gamma_0 = C\mathbb{Z}^2$. The translates of the hexagon by $\gamma \in \Gamma_0$ form a partition of $\mathbb{R}^2$ with the centers of the hexagons in the partition being the lattice points γ. Let $q = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$. The MRA is now generated by the composite dilation system $\{D_{q^j} D_{\rho^i} T_\gamma : j \in \mathbb{Z}, i = 0, 1, 2, \gamma \in \Gamma_0\}$ applied to the linear scaling function $\phi(x)$ with $\phi(v_{00}) = 1$, $\phi(v_{01}) = \phi(v_{02}) = \phi(v_{03}) = 0$ (i.e., the Bézier coefficient vector of ϕ is $(1, 0, 0, 0)$). Here, the vertex v_{00} at which ϕ has value 1 is the initial vertex of diamond boundary. The space V_j are q^{-j} dilates of V_0, i.e., $V_j = D_{q^{-j}} V_0$ $(j \in \mathbb{Z})$.

Continuous Wavelets with Composite Dilation-3

The space $V_0 \subset L^2(\mathbb{R}^2)$ consists of the linear functions $\phi_i(x)$ defined on R_i $(i = 0, 1, 2)$, with values at vertices of R_i as $\phi(v_{i0}) = 1$ and $\phi(v_{i1}) = \phi(v_{i2}) = \phi(v_{i3}) = 0$, and their translations defined on Γ_0-translates of the diamonds R_i $(i = 0, 1, 2)$. In order to describe the space V_1 we consider the original hexagon H and, within H, the smaller hexagon $q^{-1}H$, which is the disjoint union of the diamonds $R_i = \rho^i R_0$ $(i = 0, 1, 2)$ and their translations. $\Phi = [\phi_0, \phi_1, \phi_2]^T$ is refinable. The corresponding multiwavelet Ψ and the duals of the Φ and Ψ are constructed. (More details available upon request.)

Continuous Wavelets with Composite Dilation-4

$\phi_0(x)$ is refinable:

$$\phi_0(x) = \phi_2\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{\sqrt{3}}{4} \\ 0 \end{pmatrix}\right) + \frac{1}{2}\left[\phi_2\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{3\sqrt{3}}{8} \\ \frac{3}{8} \end{pmatrix}\right)\right.$$
$$\left. + \phi_2\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{3\sqrt{3}}{8} \\ -\frac{3}{8} \end{pmatrix}\right) + \phi_2\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}\right)\right] \quad (4)$$

Continuous Wavelets with Composite Dilation-5

Similarly, we have

$$\phi_1(x) = \phi_0\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{-\sqrt{3}}{8} \\ \frac{3}{8} \end{pmatrix}\right) + \frac{1}{2}\left[\phi_0\left(q^{-1}\rho^2 x + \begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}\right) + \phi_0\left(q^{-1}\rho^2 x + \begin{pmatrix} -\frac{\sqrt{3}}{4} \\ \frac{3}{4} \end{pmatrix}\right) + \phi_0\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{-3\sqrt{3}}{8} \\ \frac{3}{8} \end{pmatrix}\right)\right] \quad (5)$$

$$\phi_2(x) = \phi_1\left(q^{-1}\rho^2 x - \begin{pmatrix} \frac{\sqrt{3}}{8} \\ \frac{3}{8} \end{pmatrix}\right) + \frac{1}{2}\left[\phi_1\left(q^{-1}\rho^2 x + \begin{pmatrix} \frac{-3\sqrt{3}}{8} \\ -\frac{3}{8} \end{pmatrix}\right) + \phi_1\left(q^{-1}\rho^2 x - \begin{pmatrix} \frac{3\sqrt{3}}{4} \\ \frac{3}{4} \end{pmatrix}\right) + \phi_1\left(q^{-1}\rho^2 x - \begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}\right)\right] \quad (6)$$

Continuous Wavelets with Composite Dilation-6

In the above three expressions, the initial vertices of the supports of functions $\phi_2\left(q^{-1}\rho^2 x + \binom{\frac{\sqrt{3}}{4}}{0}\right)$, $\phi_0\left(q^{-1}\rho^2 x + \binom{\frac{-\sqrt{3}}{8}}{\frac{3}{8}}\right)$, and $\phi_1\left(q^{-1}\rho^2 x - \binom{\frac{\sqrt{3}}{8}}{\frac{3}{8}}\right)$ are the original, which is an element in Γ_0. It can be seen that all coefficients of those functions in expressions (4)-(6) are 1. Furthermore, the initial vertices of the boundaries of the supports of all other functions $\phi's$ with coefficients $1/2$ in the expressions are not in Γ_0. Those functions $\phi_i(qx + \cdot)$, which supports have no initial vertices in Γ_1, will be defined as our wavelet functions $\psi_{i,\cdot}(x)$ $(i = 0, 1, 2)$ with a certain translations. Hence, any element $\phi_i(qx + \cdot)$ in V_1 is either $\psi_i(x)$ or the difference of $\phi_j(x)$ and $\psi_j(x + \cdot)$'s, where $j = 2, 0, 1$ when $i = 0, 1, 2$, respectively.

Continuous Wavelets with Composite Dilation-7

We define the dual scaling functions $\tilde{\phi}_i(x-\gamma) = \delta_\gamma$ $(\gamma \in \Gamma_1)$, where δ_γ is the Dirac distribution at the node γ. We define dual wavelet of $\psi_{i,\sigma}$ by $\tilde{\psi}_{i,\sigma} = \delta_\sigma - (1/2)(\delta_{\sigma_1} + \delta_{\sigma_2})$, where $\sigma_1, \sigma_2 \in \Gamma_1$ and $\sigma = (\sigma_1 + \sigma_2)/2$ (we can do it because each σ is a middle point of two points in Γ_1). It can be seen that $\{\phi, \psi\}$ and $\{\tilde{\phi}, \tilde{\psi}\}$ are generalized biorthogonal in the Radon measurement. And for a function $f \in V_1$, it can be decomposed in terms of the dual bases. In order to obtain a more stable decomposition, we shall use the lifting-scheme (Sweldens [96]) to modify the bases.

C^1 Quadratic Prewavelets with Composite Dilations-1

Consider the hexagonal lattice Δ in $\mathbb{R}^2$ defined by $C\mathbb{Z}^2$ with $C = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$. Let Δ^3 be the type-3 refinement of Δ. We call $\phi \in S_2^1(\Delta^3)$ a Powell-Sabin(PS) spline or macroelement. For any $k \in \Delta$, the Hermite interpolation problem

$$\begin{pmatrix} \phi_{k,0}(\ell) & D_1\phi_{k,0}(\ell) & D_2\phi_{k,0}(\ell) \\ \phi_{k,1}(\ell) & D_1\phi_{k,1}(\ell) & D_2\phi_{k,1}(\ell) \\ \phi_{k,2}(\ell) & D_1\phi_{k,2}(\ell) & D_2\phi_{k,2}(\ell) \end{pmatrix} = \delta_{k,\ell} I$$

has a unique solution $\Phi_k = (\phi_{k,0}, \phi_{k,1}, \phi_{k,2})^T$. And $\{\Phi_{0,k} \equiv \Phi(x - \Gamma k) : k \in \mathbb{Z}^2\}$ is a basis of $S_2^1(\Delta^3)$. The BB-expressions of $\Phi_{0,i}$ $(i = 0, 1, 2)$ are given.

C^1 Quadratic Prewavelets with Composite Dilations-2

Φ is refinable with respect to the dilation matrix $D = 2I$:

$$\Phi(x) = \sum_{k\in\mathbb{Z}^2} C_k \Phi(Dx - \Gamma k), \quad x \in \mathbb{R}^2.$$

The refinement $\Delta_j := D^{-j}\Delta$ is the mid-edge subdivision that generates PS partition $\Delta_j^3 := D^{-j}\Delta^3$. The corresponding nested subspaces $V_j = S_2^1(\Delta_j^3) \subset L^2(\mathbb{R}^2)$, $j \in \mathbb{Z}$, form a MRA of multiplicity 3.

C^1 Quadratic Prewavelets with Composite Dilation-3

$$C_{1,0} = \frac{1}{12}\begin{pmatrix} 0 & 0 & 0 \\ 3 & -6 & -2\sqrt{3} \\ 3 & -6 & 2\sqrt{3} \end{pmatrix} \quad C_{1,1} = \frac{1}{12}\begin{pmatrix} 1 & -4 & 0 \\ 1 & 2 & -2\sqrt{3} \\ 4 & -4 & -4\sqrt{3} \end{pmatrix}$$

$$C_{0,0} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -\sqrt{3} \\ 1 & 1 & \sqrt{3} \end{pmatrix} \quad C_{0,-1} = \frac{1}{12}\begin{pmatrix} 1 & -4 & 0 \\ 4 & -4 & 4\sqrt{3} \\ 1 & 2 & 2\sqrt{3} \end{pmatrix}$$

$$C_{0,1} = \frac{1}{12}\begin{pmatrix} 3 & 0 & -4\sqrt{3} \\ 0 & 0 & 0 \\ 3 & 6 & -2\sqrt{3} \end{pmatrix} \quad C_{-1,0} = \frac{1}{12}\begin{pmatrix} 4 & 8 & 0 \\ 1 & 2 & -2\sqrt{3} \\ 1 & 2 & 2\sqrt{3} \end{pmatrix}$$

$$C_{-1,-1} = \frac{1}{12}\begin{pmatrix} 3 & 4 & 4\sqrt{3} \\ 3 & 6 & 2\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}$$

C^1 Quadratic Prewavelets with Composite Dilation-4

Let H be the Hermit interpolation operator, and let $H_2^r(\Omega)$ be the Sobolev space with norm $\|f\|_{r,\Omega} := \left(\sum_{|\alpha|\le r} \|D^\alpha f\|_2^2\right)^{1/2}$, where $\|\|_2$ is the L^2 norm on Ω and $D^\alpha f = (\partial^{|\alpha|}/\partial x^{\alpha_1}\partial x^{\alpha_2})f$, $\alpha = (\alpha_1, \alpha_2)$. Thus, there exists appositive constant C such that

$$\|f - Hf\|_{r,\Omega} \le C\delta^{3-r}\|f\|_{3,r}$$

for all $f \in H_2^3(\Omega)$ and $r = 0, \dots, 3$, where $\delta = |\Delta|$ the maximum of the diameters of the triangles in the triangulation Δ. Particularly, for $r = 0$, there holds $\|f - Hf\|_\infty \le C\delta^3\|f\|_{3,\Omega}$ for all $f \in H_2^3(\Omega)$.

C^1 Quadratic Prewavelets with Composite Dilation-5

Find the dual basis $\{\tilde{\Phi}_{j,k} : k \in \Delta_j\} \subset V_{j+1}$ using the matrix system

$$\langle \tilde{\Phi}_{j,k}, \Phi^T_{j,k_i} \rangle = \delta_{i,0} \|\Phi_{j,k}\|^2,$$

where $\|\Phi_{j,k}\|^2 = diag\left(\langle \Phi_{j,k}, \Phi^T_{j,k} \rangle\right)$, k_0 is the center of the Hexagon and k_i $(i = 1, 2, \ldots, 6)$ are its boundary vertices. In addition, $\tilde{\Phi}_{j,k}$ has the compact expression

$$\tilde{\Phi}_{j,k} = \sum_{i=0}^{6} P_i \Phi_{j+1,\ell_i},$$

where $\ell_0 = k_0$ and ℓ_i is the middle point of $\overline{k_0 k_i}$ $(i - 1, 2, \ldots, 6)$.

C^1 Quadratic Prewavelets with Composite Dilation-6

$\{2^{j+1}\tilde{\Phi}_{j,k} : k \in \Delta_j\} \cup \{2^{j+1}\Phi_{j+1,\ell} : \ell \in \Delta_{j+1} \setminus \Delta_j\}$ forms a Riesz basis of V_{j+1}. Since $\langle \tilde{\Phi}_{j,k}, \Phi^T_{j,\ell} \rangle = \delta_{k,\ell} \|\Phi_{j,k}\|^2$, we know the

$$\Psi_{j,\ell} := \Phi_{j+1,\ell} - \sum_{k \in \Delta_j} \langle \Phi_{j+1,\ell}, \Phi^T_{j,k} \rangle \|\Phi_{j,k}\|^{-2} \tilde{\Phi}_{j,k}$$

is in $V_{j+1} \cap V_j^T$, which yields the Riesz basis, $\{2^{j+1}\Psi_{j,\ell} : \ell \Delta_{J+1} \setminus \Delta_j\}$, of W_j, and $L^2(\mathbb{R}^2) = \oplus_{j \in \mathbb{Z}} W_j$.

Acklonolgyment

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